Contribution of the Analysis of the Mathematical Concordance to Understand the Teachers' KTMT

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Abstract

Technology is recognized by its potential to promote mathematical learning. However, achieving this potential requires the teachers to have the knowledge to integrate it properly into their practices. Several authors have intended to characterize the teachers' knowledge and developed several models, but this approach has often been criticized by its static approach, not attending neither valuing the teachers' practice. In this study we adopt the KTMT – Knowledge for Teaching Mathematics with Technology model, assuming the teachers' practice as the main scenario of analysis. We focus on the options guiding the teachers' decisions when confronted with a situation of lack of mathematical concordance while teaching functions. The situations of lack of mathematical concordance (i.e., situations where the mathematics addressed by the students is different from the one intended by the teacher) are assumed as rich and encapsulating the potential to reveal significant aspects of the teachers' KTMT. The main goal of the study is to understand what domains of the teachers' KTMT are highlighted in these circumstances. A qualitative methodology is adopted and one episode of one 10th grade teacher's practice is analyzed, based on the KTMT model. The conclusions reached show the relevance of different knowledge domains, but emphasize the Mathematics and Technology Knowledge (MTK). They also raise questions about the impact of the specific technology being used on the teachers' KTMT.

Keywords: KTMT, professional knowledge, digital technology, mathematical concordance, mathematics

1. Introduction

Teachers' knowledge has a significant impact on what is done in the classroom and, consequently, on student learning (Clark-Wilson et al., 2020; Fauskanger, 2015; Rocha, 2020a). It is the teachers' knowledge that determines the tasks proposed to the students, the learning environment created, the teachers' perception regarding the learning process, and their ability to learn from interactions with students and to adjust the initial lesson plan (Leikin & Levav-Waynberg, 2007). It is also the teachers' knowledge that guides the integration of technology into their practices and the exploration of its potential (Anabousy & Tabach, 2022; Rocha, 2020a).

Several authors have intended to characterize the teachers' knowledge and developed several models. Starting from the inspiring work of Shulman (1987) and his PCK – Pedagogical Content Knowledge construct, and going thought all the authors who used it and developed clarifications and extensions (Sevinc, in press), until the very well-known refinement of the PCK proposed by Ball et al. (2008), we have come a long way that helps us to better understand what teachers need to know to teach. The integration of technology into teachers practice has raised new challenges. Some more specific models were developed, such as the TPACK, by Misha and Koehler (2006), inspired in the PCK construct; or the PTK – Pedagogical Technology Knowledge, by Thomas and Hong (2013), and intending to promote the inclusion in a knowledge model of aspects related to instrumental genesis (Rabardel, 1995); or even the KTMT – Knowledge for Teaching Mathematics with Technology, by Rocha (2013, 2020a), a model intending to integrate the research on professional knowledge and the results from studies in technology integration.

However, the research based on knowledge models has been criticized because of the way it uses these models, often with a greater focus on distinguishing between different domains of knowledge than on their operationalization (Ruthven, 2011). Another point, addressed by Tabach and Trgalová (2019), has to do with the focus of the study.

According to the authors, most of the studies are content-driven or tool-driven. Those who are content-driven focus on the knowledge and skills needed to teach a specific content using technology (e.g., functions). Those who are tool-driven focus on the knowledge and skills needed to use a specific technology to teach mathematics (e.g., graphing calculator).

The research approach based on knowledge models has also been criticized by its focus on a static approach (Tabach, 2011), not valuing the dynamic character of the teachers' professional development and, specifically, of their knowledge. Some of the studies tend to address the teachers' knowledge disconnected from their practice, without taking into account the difference that may exist between a task (what it presupposes) and its implementation. Here we base the analysis of the teachers' knowledge on what happened in the classroom, and we use the model KTMT to identify aspects of the teachers' knowledge intending to understand the teachers' needs for professional development.

The focus of the investigation is centered on the characterization of the teachers' knowledge. It is intended to characterize the teachers' knowledge from their practice, showing how this practice contributes to the dynamism of the teacher's professional development. The study focuses on one mathematical content (functions) using a specific technology (graphing calculator), adopting both, a content-driven and a tool-driven approach. A circumstance that raised some interesting questions about the characteristics of the teachers' knowledge that we have not seen addressed in other studies.

The study is based in the options guiding the teachers' decisions when confronted with a situation of lack of mathematical concordance. It assumes the unexpected situation of lack of mathematical concordance as a rich situation encapsulating the potential to reveal significant aspects of the teachers' KTMT; somehow in line with the *hiccup* construct by Clark-Wilson and Noss (2015), defined as "the perturbation experienced by a teacher during teaching that has been triggered by the use of mathematical technology" (p.92), or the *pivotal teaching moment* construct by Stockero and Zoest (2013), defined as "an instance in a classroom lesson in which an interruption in the flow of the lesson provides the teacher an opportunity to modify instruction in order to extend or change the nature of students' mathematical understanding" (p.127). In both cases, the potential of unexpected situations in the classroom is recognized, as it is in this study. The main goal here is to understand and discuss what the teachers' decisions, when confronted with situations of lack of mathematical concordance, can tell us about their KTMT. Specifically, it intends to understand:

What domains of the teachers' KTMT are highlighted by a situation of lack of mathematical concordance?

2. Mathematical Concordance

Mathematical concordance refers to the level of alignment between any of the mathematical fields that interact in the learning process, which includes the mathematics of the teacher, the mathematics of the student, the written mathematics (as the one in a task or in the textbook) and the mathematics of the technology (Zbiek et al., 2007). This construct is different from the one of mathematical fidelity, since, unlike the latter, it does not respect to a concordance with something pre-established (such as the mathematics' science). It is possible to find different levels of mathematical concordance, being these levels seen as a continuum.

Some studies that have devoted attention to mathematical concordance have focused on the alignment between the teachers' intentions and the mathematics worked on by the students, a perspective that we also adopt in this study. Among the situations usually addressed in the literature, the ones in which there is a low level of concordance between the mathematics in which the students are involved, and the mathematics the teacher intended them to be involved when designing the task, are particularly referred to.

Dugdale (2008) was one of the authors who came across this low level associated with the well-known game of green globes. The objective of this game is to create functions whose graphical representation passes through green globes randomly distributed by a cartesian referential. The globes intersected by the function explode and the player's goal is to make all of them explode using the smallest number of functions. From a mathematical point of view, the game's goal is to make students think about different families of functions. However, what the author found was that students subverted this goal discovering that a function like $f(x) = 10 \sin(10x)$, taking into account its periodicity, would intersect any globe. In these circumstances, the concordance between the mathematics that the task intended to address and the one addressed by the students became extremely low. Still, the little mathematics involved was related to what the teacher initially intended, something that does not always happen.

The example presented here refers to a situation where mathematical concordance is very low and, as so, a situation usually noticeable by the teacher. However, this is not always the case, as referred by Zbiek et al. (2007), and cases may arise in which the teachers think that learning is taking place about contents that in fact are not even being worked by the students. And this is the main reason why Sutherland and Balacheff (1999) consider this notion to be important and why they understand that teachers must be aware of it. According to them, although situations of lack of mathematical concordance can occur under any circumstances (regardless of the technology being used or not), when technology is being used, this occurrence can happen more often and in situations where the teachers are not aware of them.

This propensity for the lack of mathematical concordance to be more difficult to detect when technology is involved has to do with the fact that it provides a rich environment, where different strategies can often be adopted. The exploration of this diversity leaves students with greater freedom, which may allow them to find approaches not anticipated by the teachers and end up avoiding the work with mathematics that the teachers intended (Hoyles & Noss, 1992). And the authors consider these circumstances somewhat paradoxical because, on the one hand, there is potentially greater mathematical richness, as students must ponder mathematics and make their own choices; but on the other hand, students may be able to find a strategy mathematically poor. This does not necessarily mean that the students' activity becomes empty of mathematical learning whenever there is a situation of lack of mathematical concordance. What often happens is that the mathematics content worked on by students is not the one the teachers intended to address at that time. These are thus situations that the teachers must be aware of and whose knowledge can be potentially enriching due to its revealing character in relation to the way students approach the tasks proposed. However, these situations can also be rich to provide information about the teachers' professional knowledge.

3. Knowledge for Teaching Mathematics with Technology - KTMT

Several authors have sought to develop models to characterize teachers' knowledge associated with the use of technology. Among these, the TPACK model, by Mishra and Koehler (2006), deserves to be highlighted, due to the great attention it has received in many studies (Mailizar et al., 2021). Even so, this is a model that does not take into account all the results that have been achieved in countless studies that have focused on technology integration, which in some way can be considered a limitation. However, the basic idea of the model, of some mutual influence between knowledge domains (already present in the model by Hill and Ball (2009)) has been widely recognized. And these are the central assumptions that led to the development of the KTMT model, which is the basis for this work.

KTMT can be seen as grounded in a set of base domains: knowledge of Mathematics, curriculum, teaching and learning and technology. The knowledge of the curriculum is, however, considered in a transversal way, on the one hand influential on other domains and, on the other hand, comprehensive, including the beliefs and conceptions of the teacher, as well as aspects related to the context in which the teacher is inserted. Knowledge of technology involves the capabilities needed to operate with a given technology and basically consists of knowing how it works, that is, what it does and how it does it (from an operational point of view). A conceptualization in line with the one present in other models. The remaining base domains considered are conceptualized in line with what can be found in other knowledge models. The fusion between the domains relating to students and their learning processes and the teaching process is not intended to constitute a real change. It is based only on the intention of maintaining some simplicity and on the recognition of the existence of mutual influences between the way in which the teachers conceptualize student learning and the teaching options they assume.

This model, as already mentioned, also intends to recognize the importance of knowledge developed at the confluence of more than one domain, as the schematic illustration in Figure 1 intends to show. Hill and Ball (2009) mention the importance of knowledge that goes beyond a single domain, referring, namely, to Knowledge of Mathematics and Teaching and to Knowledge of Mathematics and Students. Mishra and Koehler (2006) also recognize the importance of this type of knowledge, giving it great relevance in the model they developed. However, the difficulties that Cox and Graham (2009) describe in detail when trying to rigorously characterize this knowledge cast doubts on the usefulness that a model of increasing complexity may have and suggest that this may not be the best approach. However, an analysis of the existing literature indicates a set of knowledge associated with the use of technology that goes beyond knowledge in each of the base domains, and which seems to be necessary for the development of knowledge to teach Mathematics with technology.

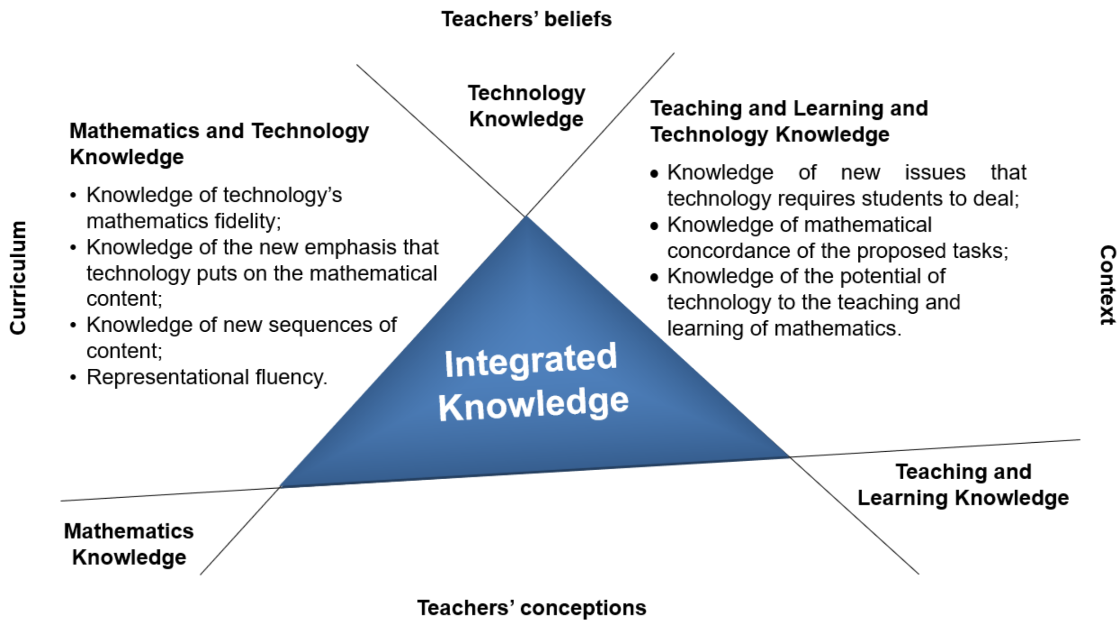

Figure 1. KTMT Model (Rocha, 2020a, 2022)

As so, KTMT, in addition to knowledge in each of the base domains, encompasses two sets of inter-domain knowledge:

1) Mathematics and Technology Knowledge – MTK

This is knowledge that focuses on how technology influences Mathematics, emphasizing or limiting certain aspects and which necessarily includes

- Knowledge of the mathematical fidelity of technology, that is, knowledge of the level of agreement between a mathematical result and the result presented by a technology (Dick, 2008). And this fidelity may not occur due to a discrepancy between the technology and the conventional mathematical syntax (for example, when the use of parentheses required by the technology does not coincide with the one of Mathematics) and due to limitations caused by the representation of continuous phenomena through discrete structures (for example, when plotting a trigonometric function with a very small period in a small screen).
- Knowledge of the new emphases that technology places on mathematical content (for example, encouraging more intuitive approaches or requiring a different domain of the influence of the values represented in the coordinate axes on the appearance of the displayed graph) (Rocha, 2016, 2020b; Roorda et al., 2016; Zbiek et al., 2007).
- Knowledge of new content orders (Bosley et al., 2007; Drijvers, 2013).
- Representational fluency, involving knowledge of how to move between representations and the options to be taken to better illustrate or support statements (Rocha, 2016; Zbiek et al., 2007).

2) Teaching and Learning and Technology Knowledge – TLTK

This is knowledge that focuses on how technology affects the teaching and learning process, enhancing or limiting certain approaches and that necessarily includes

- Knowledge of the new questions that technology poses to students (such as the search for a suitable viewing window (Rocha, 2020b))
- Knowledge of the mathematical concordance of the proposed tasks, that is, the alignment between the teachers' intentions and the mathematics effectively worked on by the students (Zbiek et al., 2007).
- Knowledge of how to articulate different types of work, taking advantage of the potential offered by technology and giving the student a more active role in learning (Dunham, 2000).

Finally, KTMT includes Integrated Knowledge – IK. This is a knowledge that simultaneously articulates knowledge in each of the base domains and in the two sets of inter-domain knowledge.

4. Methodology and Study Context

Given the objective of this study, we opted for a qualitative and interpretive methodology based on the analysis of one teacher's practice (Yin, 2003). We will designate the participating teacher by Carolina. The criteria for choosing this teacher focus on the teacher's experience – it has to be an experienced teacher (20 or more years as a teacher) -, on the teacher's attitude toward the use of technology to teach mathematics – it has to be a teacher valuing the use of technology (have been involved in projects and similar iniciatives related to technology integration in the classroom) -, experience with this technology – it has to be a teacher with a limited experience in using graphing calculators with students – less then three years using graphing claculators with students. Carolina is a teacher with more than thirty years of professional experience, enthusiastic about the use of computers in the teaching of Mathematics (as described by herself, in the first interview of this study). She has, however, no experience in using the graphing calculator, this being the second time she has used it with 10th grade students (16 years old). She never had any training related to the graphing calculator use, having learn on her own what she knows about this technology. And in her opinion, she does not know much about how to operate the machine, although she believes she knows enough to use it with her students.

Data collection included, in addition to an initial interview with the intention of getting to know the teacher and her opinions, observation of a wide range of lessons (14 lessons) and interviews intending to understand the teacher's point of view about the lessons, before and after each class (interviews taking from 20 to 60 minutes, where the teacher shared freely her expectations/reflections). All interviews and all observed classes were audio-recorded.

The tasks proposed by the teacher to the students were assumed as the unit of analysis. For each task the mathematics intended by the teacher and the mathematics worked by the students was analyzed. This analysis allowed the identification of one episode where differences occurred, i.e., where a situation of lack of mathematical concordance occurred. This episode occurred in the first class devoted to the study of how to solve absolute value inequalities and the students were using the graphing calculator TI-83 Plus (Note 1). The episode was then analyzed based on the KTMT model, identifying moments in which different knowledge domains were involved. A special attention was given to the teacher's Technology Knowledge (TK), Mathematics and Technology Knowledge (MTK), and Teaching and Learning and Technology Knowledge (TLTK). It was namely identified what happened in the classroom and how that resulted from the teacher's TK, as well as the impact of the situation of lack of mathematical concordance in the teaching approach assumed by the teacher (revealing her TLTK) and in the mathematics being addressed (revealing her MTK).

Table 1. Categories of Analysis and Their Relation to KTMT Model

Category of analysis	Knowledge domain
Teachers' knowledge of technical aspects related to the situation of lack of mathematical concordance	Technology knowledge (TK)
Teachers' knowledge of the impact of the situation of lack of mathematical concordance in the mathematics being addressed	Mathematics and Technology Knowledge (MTK)
Teachers' approach to the situation from a mathematical point of view	
Teachers' expectations about the students' reaction to the situation	Teaching and Learning and Technology Knowledge (TLTK)
Teachers' approach to the situation from a teaching and learning point of view	

During the study, it was given attention to ethical issues, namely ensuring the annonimity of the teacher and, in relation to that, ensuring the audio records will be used exclusively by the reasearcher. There was also a concern that the teacher was aware, from the beginning of the study, about what data will be collected and how they will be collected. The options related to data collection were discussed with the teacher intending to have her concordance. After the data analysis, the results achieved were shared with the teacher for validation. She expressed agreement with the interpretation and conclusions produced.

5. Results

5.1 The Teacher – Carolina

Usually, Carolina's proposals to the students included some guidance on what to do with the graphing calculator and they followed what was suggested to them. Still, situations sometimes arose where they made a different use of technology. Among these situations, some were equally adequate approaches that did not significantly change what the teacher had thought; others were less adequate approaches and do not allow the students to conclude the task; and still others were completely different approaches from what had been thought by the teacher, but allowed them to obtain the requested answer. In the latter case, are the situations of lack of mathematical concordance. Here we analyze the episode identified based on the analysis of Carolina's lessons. This is an episode in which the approach chosen by the students allows reaching the desired result, but it differs from what the teacher had thought, not addressing the mathematical content as she had thought.

This episode took place in a lesson dedicated to solving inequalities with absolute value, i.e., inequalities such as $|2x - 4| \leq 2$. Carolina's intention was to privilege the graphical approach, leaving the analytical work to the end of the lesson. The idea was to use the graphing calculator to represent an absolute value function (e.g., $y_1=|2x-4|$) and a constant function (e.g., $y_2=2$) and then solve the inequality using the points of intersection of these two functions graphs (see figure 2), which would be determined by the machine (T: Teacher):

> T - I am planning to start by a graphical approach. So, with the calculator, look for… let's say, the absolute value of something smaller than 2. Using the calculator, we have the absolute value function, we have y=2, we look for the intersection points. So, what do we want? We want the absolute value function to be smaller [than 2]... So, do this graphical approach with the calculator and then move on to the analytical part. (pre-lesson 9)

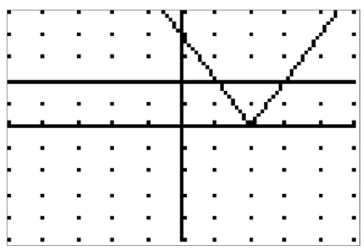

Figure 2. Graphical Approach to Solve the Inequality $|2x-4| \leq 2$

The teacher was building on her knowledge of the potential of technology to teach mathematics (TLTK) and planning to use the technology to start with a graphical approach, which she implicitly seems to think is beneficial to students. This option also presupposed a recognition of how working with different representations can be important to develop students' mathematical knowledge and how technology allows it (MTK).

However, in the classroom, the use that several students made of the technology differed from what Carolina had anticipated, as they introduced the entire inequality in y1 and not a part in y1 and another in y2. The graph displayed by the machine seemed to surprise the students (see figure 3) who reacted asking for help:

> Student 1 – Teacher, can you please come here?
>
> Student 2 - Teacher, this gives something weird.
>
> Student 3 - Mine too. (lesson 9)

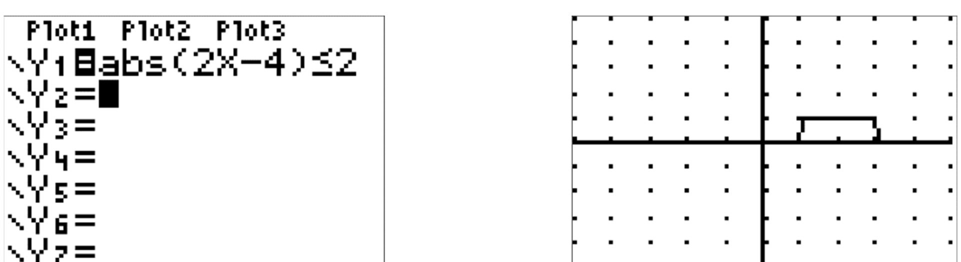

Figure 3. The Students' Approach to Solve $|2x-4| \leq 2$

Carolina had not anticipated the approach followed by the students and had not yet realized how the calculator dealt with it:

> T - They put the inequality right away, so that was right there… the solution set. I had not seen that it would work. (…) If I knew it was like that, maybe I'd try a different approach… Maybe I did it by hand on the board and… I mean, I think that's not funny. The solution shows up right there like that, they don't even have to think about anything. (post-lesson 9)

Upon seeing the use that the students have made of technology, Carolina realized that this allows them to immediately obtain the intended answer, with no need to mobilize their mathematical knowledge. She was thus faced with a lack of mathematical concordance. The mathematics worked on by the students was not exactly what she had in mind. And, according to Carolina, it would be no mathematics at all.

The teacher noticed that the students had no idea of what they are looking at, not realizing that the machine is immediately showing them the solution set, and she said it spontaneously when talking about the class:

> T - Afterwards they also didn't know how to interpret what they saw. They didn't understand what they are seeing. (post-lesson 9)

The students' inability to understand what they are seeing is identified by Carolina. Nevertheless, she did not realize that this might be due to the fact that a proper reading of the information offered by the technology requires mathematical knowledge. The feeling that the mathematical content she intended is not being addressed and, more than that, the idea that no mathematical content is being addressed, prevailed. She then sought to manage the class to ensure mathematical concordance between what she intended and what students did:

> T - Ok. So, tell me what you think. How did the calculator solve the problem? Or how would you solve the problem if the calculator didn't do it for you? Let's assume… Your calculator right now only has the ability to plot graphs of functions, nothing else. That's it, and you must do the rest… (lesson 9)

She chose to direct the work of the students, saying that they can only use the graphical capability of the technology to draw graphs of functions. Despite realizing that the students had no idea that the technology was immediately giving them the solution, she mentioned this to the students and somehow forbade its use. Faced with something unexpected and that her professional knowledge (KTMT) does not support her to integrate, she thus exhibited an uncomfortable reaction towards the integration of the technology at that time. And after solving the inequality graphically, the way she intended, Carolina reversed what she had planned for the lesson and focused on the analytical resolution of inequalities. As she stated:

> T - I had thought to solve more inequalities graphically. The idea was to ask them to use the calculator, to draw the graphs, but then I thought… I mean, if they're going to put this in the machine, the draws come out there and that's it. So, I didn't insist as much as I had thought to insist. (post-lesson 9)

Not knowing how to take advantage of this feature of the technology's operation, she opted to exclude it. Later, she justified this change with the fear of taking more time and no longer being able to address everything she intended to in this lesson, also expressing concern about the possibility that the students might end up getting confused:

> T - I had set the point where I wanted to be. And since I hadn't thought of a strategy to… to take advantage of that… I was afraid that at the end I wouldn't be able to finish where I want, you know? (…) And I think I wouldn't be able to make the best of it… (…) And at the end they might get confused… Those heads get confused very easily, you know? That's why I decided not to take those unknown lands… I'm a crab, crabs walk on land. (I laugh) It's true! (she laughs too) (post-lesson 9)

Even so, the main reason for the inflection carried out in the classroom seems to be related to a failure to associate this potentiality of the calculator with any positive contribution to the students learning. I.e., she was unable to integrate the technology and the mathematics (MTK). Indeed, when I asked her what contributions, positive or negative, she considered the calculator brought to this lesson, the reference to the machine's presentation of the solution-set of inequalities was immediate:

> T - Giving the answer right away, for me, is negative. It's negative because I think that's it, they don't understand anything, they don't know what they are doing, but they get the right answer. (post-lesson 9)

She understands, however, that this is a feature of the calculator that she will have to deal with, and she expressed the intention to reflect on the question and to come back to it later (R: Researcher, T: Teacher):

R - Basically what you think right now, doesn't mean that later… you can think and you can change your mind… but right now what you think is that you will do what you can to get around and prevent students from using this or that in the next lesson, or in a future lesson, you will address this and somehow (interrupts me)

T - No no, I want to get back to this. I mean, now I'm going to try it at home, right? (…) I mean, the machine does this, and you cannot pretend it doesn't. (post-lesson 9)

She recognized that this, being a feature of the technology, is something she had to find a way to deal with. She clearly expressed the need to reflect and implicitly to develop her professional knowledge (KTMT) to integrate this situation.

In the end, however, she did not address the issue. At the time, we were at the end of the 2nd term and some classes were cancelled because the students were involved in some sport activities in the school. Carolina, who was already concerned about fulfilling the teaching of all the syllabus contents, felt even more pressured and this was not considered a priority issue. By the end of the study of the topic, the students graphically solved many inequalities in the classroom. They always did this by introducing two functions and determining the points of intersection of the graphs, as the teacher told them to do. The introduction of an inequality in the calculator, directly into the space dedicated to functions, did not happen again.

5.2 Analysis of the Implicit KTMT

The situation of lack of mathematical concordance has its origin in the lack of knowledge of the teacher regarding this aspect of the operation of the technology (TK). Confronted with the situation, the teacher is afraid that the students start to use an automatic technology procedure, without developing the mathematical knowledge to deal with the situation autonomously. So, her main concern is related to the impact of the technology on the students mathematical learning (MTK). However, although starting from a lack of teacher's technology knowledge (TK), the episode of lack of mathematical concordance stems from the impossibility of the teacher to anticipate the students use of technology (TLTK).

To Carolina, y= is a place to introduce the expression of a function but, for the students, it is a place to introduce information. This difference has its origin in the knowledge of the mathematics, now interconnected with the technology (MKT). y equal to some expression defines a function (together with its domain and arrival set). This mathematical knowledge (MK) prevented the teacher from imagining the students' approach, and her reduced experience using graphing calculators with students did not help her to anticipated what happened (i.e., her limited TLTK). However, this teacher has a positive attitude towards the use of technology to teach and learn Mathematics. When confronted with the unexpected use of technology done by the students, she assumes she must find a way to integrate it. Nevertheless, in the classroom she cannot find a way to do it. In the moment, she only can see a use in the style of "black box" and she does not see any contribution from that to her students mathematical learning. The disorientation provoked by the unexpected use of the technology done by the students, does not allow her to see this episode as an opportunity to discuss and deepen the students' knowledge related to the difference between functions and conditions. In these circumstances, it is her difficulty in identifying how to use the technology to promote mathematical learning, that takes her to the decision of avoiding the use of the technology's feature discovered by the students. I.e., the teacher's mathematical and technology knowledge (MTK) is determinant for the decisions assumed in the rest of the lesson.

6. Conclusion

The conclusions of this study suggest that a situation of lack of mathematical concordance is related to the knowledge of technology (TK) and to an unexpected approach of the students and, as so, is related to the teacher's knowledge of the students and the technology, a part of her TLTK. However, the knowledge that guides the teacher's decisions to deal with the situation of lack of mathematical concordance is the teacher's mathematical and technological knowledge (MTK). It is the teacher's concern about the mathematics the students will learn at the end that guides all her decisions. And this is a conclusion contradicting results achieved in other studies, where the teachers' TLTK was determinant to their choices and decisions, although in that case the participants of the study were pre-service teachers (Rocha, 2022). However, it suggests that teachers tend to focus more on their MTK or on their TLTK, and not on both. We can then conclude that analyzing the specific characteristics of this knowledge guiding the teacher's decisions, can be very important to understand how the teacher's knowledge can be improved to achieve a better integration of the technology. And Tabach and Trgalová (2019) state the need to develop teachers' knowledge to achieve a deeper integration of technology. The results of this study suggest that even teachers who value the use of technology to teach and learn mathematics, can face difficulties to think about ways of using the features of the technology to teach mathematics. As also emphasized by Bozkurt and Uygan (2020), this study provides evidence about how technology requires some flexibility to adopt approaches that differ from the ones usually taken, an issue related to the need for professional development focused not only on the use of technology, but mainly on the use of technology to teach mathematics, as emphasized by several authors, such as Faggiano et al. (2021). The point highlighted by this study is the fact that the professional development is need even for teachers with experience in the use of computers (as was the case of this teacher). Meaning that the differences between technologies can be relevant for their integration in teachers' practices. And this is a question addressed by Rocha (2022) in relation to pre-service teachers, but it is a question needing further attention.

The results of this study also point to another interesting aspect related to technology integration. This teacher tried to use the technology as a mean to get a fast access to graphs of several functions. But besides this use, the approach followed was basically the traditional approach (meaning the one that would be done without technology access), a common situation, as pointed by Bozkurt and Uygan (2020). This intention to keep the usual way of addressing the mathematical content, prevented the teacher from identifying possible ways of taking advantage of this potentiality of the technology and somehow seems to be an obstacle for its integration in the practice.

However, the main contribution of this study is the evidence provided of how the analysis of situations of lack of mathematical concordance can offer rich information about the teachers' knowledge. This kind of situations do not occur often. In this study, only one episode of lack of mathematical concordance was observed in a set of 14 lessons observed. Nevertheless, this episode offers evidence about aspects of the teacher's knowledge and about the teacher's technology integration into her practice. The potential of this unexpected moments in the classroom has already been recognized by other authors (such as Clark-Wilson and Noss (2015) and their notion of *hiccup*, and Stockero and Zoest (2013) and their notion of *pivotal teaching moment*), however, this potential has never been studied as a way to access and characterize teachers' knowledge.

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Note

Note 1. In some more recent models of calculators the situation described here could not occur because the technology classifies it as a syntax error.