Estimating the Tail Index of Conditional Distribution of Asset Returns

Feruzbek Davletov[1]

[1] Center for Economic Research and Reforms, Republic of Uzbekistan

Correspondence: Feruzbek Davletov, Center for Economic Research and Reforms, Republic of Uzbekistan.

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Abstract

Massive stock market failures in the past decades cast a doubt on the standard normality assumption of many economic models. Despite decent research on the non-Gaussian characteristics of many financial time series, the question of tail heaviness still remains open. We conduct diagnostic analysis on the conditional distribution of asset returns of small/large companies (Russell 2000 and S&P 500) to look for clear evidence on the presence of heavy tails. We employ extreme value (EVT) tools in order to estimate the shape parameter ($\hat{\alpha}$) of Generalized Pareto distribution (GPD) using a well-known "Hill estimator". It turns out that the shape parameter lies in the interval $\hat{\alpha} \in [2.6; 2.8]$ implying that the conditional distribution of asset returns supposedly has finite mean and variance. We also find an evidence that the tail estimates experience structural breaks during 2008 Global Financial Crisis.

Keywords: conditional distribution of asset returns, heavy tails, extreme value tools, generalized pareto distribution, Hill estimator, tail index

JEL: C01, C46, C58.

1. Introduction

Imagine predicting the unpredictable or expecting the unexpected. Our lives would lose its taste and meaning. It is the unpredictable which makes humanity to outdo itself, leading to great achievements throughout the history. "Black Swans" in the literature are known as rare and unpredictable events with serious ramifications, which are not normally expected. The famous book of Taleb (2007) "Black Swans" inspired me to express the idea of extremal events from the perspective of economics. In the last few decades, we have witnessed some serious economic crises, which are directly reflected in stock market indicators. It is obvious that many economic models fail to predict such abnormalities in the world, the standard assumption in which is the normality of distribution under the consideration. For instance, the option pricing models require the normality of asset returns or take any other models in risk management. It is a well-known fact that the empirical unconditional distribution of most of the financial time series possess fat tails (Mandlebort, 1969). In spite of the fact that Gaussian or in other words Normal distribution has attractive limiting properties, it might lead to erroneous results when the data is not modelled with heavy tails. The difference between the two is that Normal distribution does not give enough weight to extreme observations in the tails. Or in other words, the fat tailed distribution's tails decay slowly, so that they are not exponentially bounded.

This paper has two goals. The first one being concerned with whether the conditional distribution of asset returns is heavy tailed and then to quantify the corresponding tail characteristics. Following this, the second objective is to check this tail behaviour over time for the existence of structural breaks. In order to do conduct our research, we look at the financial asset returns of S&P 500 and Russell 2000 that are composite stock indexes, which represent large and small companies respectively. The choice of our sample is motivated by the hypothesis that companies with a huge gap in terms of market capitalization tend to have different tail characteristics (Cenezoglu, 2008). Consequently, these indexes are modelled with GARCH (1;1) that introduced by Bollerslev (1986) to capture the volatility clustering. Then we obtain the corresponding standardized residuals. Family of GARCH models assume the normal distribution for innovations that in fact have the same shape as the distribution of conditional returns. As a consequence of this, various diagnostic tests to check the normality assumption are carried out on standardized residuals. Therefore, after collecting preliminary evidence for the existence of heavy tails, we embark on the evaluation of tails without making any assumptions about the underlying distribution. Generally, the behaviour of a tail is characterized in the so-called tail index α, which is positive $\alpha > 0$ for the class of heavy tailed distributions. Tail index is ambiguous in terms of interpretation. For instance, it implies that the lower the index the heavier the tail, and it also tells the number of finite moments of a distribution (Cont, 2001). Going further, we apply one of the branches of extreme value theory (EVT), mainly the peak over threshold (POT) approach in order to estimate the tail index. Literature provides with wide variety of tail index estimators, so the choice is given to the popular “Hill estimator because of its attractive asymptotic characteristics and computational simplicity. However, it requires a choice of the number of extreme observations in the tails. As a solution to this, we construct so called “Hill Plot” that plots Hill estimates against different number of upper order statistics, which is mainly the sequence of observations in an ascending order. Then, one can infer the estimated tail index from the stabilized area of the graph. In cases, when the “Hill plot” is uninformative, we use the methods proposed by Dree et al. (2000) and Resnic and Starica (1997). These techniques help to remove the volatility of the graph and give more weight to small order statistics, so that they appear to be more distinguishable in the plot. Analysis are done with respect to both left and right tails of conditional distribution of asset returns based on the stylized fact of profit and loss asymmetry in returns. Diagnostic analysis shows clear evidence of tail heaviness so that the tails can be modelled by Generalized Pareto distribution (GPD). The results in Table 5 present that the tail index estimates can be summarized in the interval $\hat{\alpha} \in [2.6; 2.8]$, which are based on the smoothed version of the “Hill plots”. The implication is that the underlying distribution has finite first and second moments, but infinite upper moments such as kurtosis.

In the next step, we are interested in the tail index constancy over time, since it has huge implications for the financial risk management and forecasting. Mainly, we suspect that tails become heavier after the 2008 Global Financial crisis. It implies that the probability of extreme events increases in the tails of conditional asset returns. In order to check this hypothesis, we refer to the test proposed by Quintos et. al (2001) for unequal sample size. This test is an extension from the work of Loretan and Phillips (1994), which has a null of tail constancy. Overall, we can state two research questions:

1. Whether conditional distribution of asset returns possess heavy tails, and if yes how heavy are they?
2. Whether the tail index, which represents the tail behaviour, is constant over time?

The paper is planned as follows. First, we introduce the theoretical background of our topic. Then, we shift to the data analysis, which is followed by the discussion of methods that are applied in throughout the paper. Finally, we report the results and give a conclusion about our findings and their implications.

2. Literature Review

One of the pioneers to discuss the distributional properties of financial data was Mandlebort (1960, 1963, 1969), who argues that empirical distributions of asset price changes are too “peaked” around the mean with extraordinarily long tails and proposes a family of stable Paretian distributions, which is primarily defined by location, scale and shape parameters. Stable distributions have a parameter, which defines the tail behaviour, called the tail index, the stability index or the ‘tail shape’ parameter alpha (α), which is between zero and two for a distribution to be stable. So that the less the index the heavy the tail, whereas the boundary $\alpha = 2$ corresponds to a normal distribution. In contrast, vast majority of papers rejected the stable models with $\alpha > 2$ (DuMouchel 1983, Jansen and De Vries 1991, Loretan and Phillips 1994). However, McCulloch (1997) demonstrates the invalidity of their findings. Since then, it has been empirically tested and found that asset returns are leptokurtic, or that they have observations peaked around the around the mean and heavy tails (Fama 1965, Press 1975, Buckle 1995, Eberlein and Keller 1995). This high “tailedness”, also known as kurtosis, does not converge to 3 in the case of normal random variable even in extensive periods. Literature provides some models that can successfully explain the excess fourth moment in the financial time series. For instance, Engle (1982) introduces autoregressive conditional heteroscedastic (ARCH) processes with varying conditional variance, which capture the presence of non-linearities in the data or the fact of tail heaviness. In these processes, one period forecast variance is explained by the past information. Or in other words, the universal feature of asset returns is that large returns are expected to be followed by large returns, of either sign, or small returns tend to be followed by small returns. Then, Bollerslev (1986) generalises it by extending the lag structure (GARCH), where he models the return as an innovation times variance conditional on past volatility and returns. Generalized autoregressive conditionally heteroskedastic models (GARCH) allow to capture this phenomenon because it is parsimonious, since it allows conditional variance to be affected by the infinite past. However, the disadvantage of such models is that they are still unable to explain the extreme outlier activity in the tails of distribution. It is a fact that, GARCH models assume the normality of innovations and that the shape of the distribution of innovations is the same as the shape of conditional distribution of returns. Thus, any mistake in modeling the distribution of innovations might cause inaccurate assessment of future risks or mispricing of options. Therefore, McNeil and Frey (2000) demonstrate the efficiency of GARCH processes for financial risk management, when they are modelled with fat tailed innovations. Generally, the distributions of GARCH innovations are subject to proper diagnostics, which is based on estimated innovations. The literature on analysis of GARCH residuals suggests a three-step plan or so-called Backing-out approach (McNeil and Frey, 2000). First, one gets the maximum likelihood estimates of GARCH coefficients and residuals. Then, graphical tools such as Q-Q plots could be applied in order to determine whether innovations match normal distribution. And finally, if you get enough evidence on the existence of fat tails or the evidence that data comes from Paretian distributions, shape parameter of the tail could be estimated. The main reason of using stable law distributions was their attractive central limit property, but this property is not persuasive when $1.5 < \alpha < 2$ because of slow convergence of "convolutions" to their limit (DuMouchel 1983). Thus, the idea "let the tails speak for themselves" comes to light, which gives a rise to a family of tail index estimators that only utilize extreme observations instead of using the whole distribution. Thus, it does not require the existence of fourth moment. Pickands III (1975) uses percentile method to estimate the tail index, which was proven to be strongly consistent and asymptotically normal by Dekkers and De Haan (1989). In this manner, Hill (1975) proposes an estimator, which is consistent (Mason, 1982) and asymptotically normal (Hall, 1982) and depends on the choice of extreme order statistics from the sample. Furthermore, Dekkers et al. (1989) present an extension of Hill's estimator for high quantile and endpoint estimations. A very important contribution to the comparison of above estimators was made by De Haan and Peng (1998). They calculate the asymptotic mean squared errors for each estimator and conclude that no estimator dominates another, unless one determines the optimal choice of threshold and convergence parameters. Threshold selection creates trade-off between bias and variance. Therefore, literature provides various ways of doing it. For instance, the standard method is to graph the values of Hill estimator against the number of upper order statistics and then depict the value of shape parameter from the stable region in the plot. But above methods are called heuristic or non-parametric. Another way of threshold selection is to choose a fixed fraction of sample such as 10% or 5%, otherwise one might end up including too many observations in the tails, which would lead to the biased estimates (Loretan and Phillips, 1994; DuMouchel, 1983). Moreover, other methods include the minimization of the mean squared error of the estimator, which is usually done by bootstrapping and beyond the scope of our paper (Hall, 1990; Danielsson et al., 2001). On contrast, DuMouchel (1983) suggests choosing some fixed fraction of the sample (5%, 10% etc) and defines it as 'a compromise between the practical need for enough observations to be included in the estimation and theoretical desire to describe the distribution'.

One aspect of tail behaviour whether tail changes over time, implying that the probability of extreme market movements rise or decrease, is also important for risk assessment and modelling. There is empirical evidence that the tail index of some financial data is time varying. For instance, Phillips-Loretan (1990) and Koedijk et al. (1990) create tests, which use "Hill estimator" and presume the break-date is known, and reject the null hypothesis that tail fatness doesn't change over time, for the cases of Japan and Western European countries by applying exchange rate data. Pagan and Schwert (1990a,b) also reject the tail constancy by applying tests, which require stricter moment conditions. Great contribution is done by Quintos and Phillips (2001), who also construct recursive, rolling and sequential test by using "Hill estimator" for the constancy of tail over time where the breakpoint is unknown and apply it for daily stock prices during Asian financial crisis. However, since the "Hill estimator" is based on the k largest order statistics. Interestingly, Quintos and Phillips (2001) show that DuMouchel's (1983) rule about fixed threshold selection, that k should be a fixed fraction of sample, leads to wrong test sizes, since that fixed fraction increases rapidly as sample size rises leading to divergence. The importance of tail variation over time lay in the Value at Risk (VaR) estimations for the riskiness of loss for investments. Bollerslev and Todorov (2014) emphasize the importance of tail change over time because it facilitates to grasp the aggregate returns in the market and their cross-sectional differences.

Overall, the literature provides evidence on the fat tailness of many financial times series. In addition to that, we observe a number of studies confirming the existence of structural breaks in the financial times series, implying that the tail characteristics change over time. However, the literature lacks the complex analysis of asset returns of small and large companies, with expected different tail risks, tail constancy over time and proper application of superior types of Hill estimator.

3. Data

The dataset contains two composite indexes, mainly the S&P 500 and Russell 2000, which represent 500 largest and 2000 small (Note 1) companies respectively. This dataset is available from Yahoo database (MSFT, 2019). It is important to choose a span so that market turnovers with severe consequences are included. Because of this, the length has been chosen depending on the availability and includes different stock market crashes such as 1987 "Black Monday", Asian crisis and Global Financial crisis. Also, the choice of this particular dataset is motivated by the interesting fact that small and large companies are expected to have different tail risks since they react differently to unexpected changes in the market (Cenesizoglu, 2008). Our sample consists of the daily closing prices of stock indexes, so that the returns are calculated as continuously compounded

$$X_t = \log(P_t) - \log(P_{t-1})$$

where X_t is a log return for time t and P_t is the respective price for time t. The data ranges from 09/09/1987 to 07/18/2019 for the full sample. In order to check the existence of structural breaks in our time series during the 2008 Global Financial crisis, we divide our sample into two sub-samples, where they range from 11st January 1988 (Note 2) to 31st August 2008 and from 1st September 2008 to 18th July 2019 respectively. Table 1 gives the summary statistics for full and sub samples. The starting point is to analyse the raw data and look for the preliminary tendencies for tail heaviness. To start with, one notices that in absolute values the minimum return is lower than maximum return for large caps and the opposite for small caps. Yet this might indicate the distributional imbalances. The biggest losses and gains (Note 3) in the full sample happened during the times of Global financial crisis for both stock indices, specifically the interval includes the October and November 2008. All this implies the immensely unstable phase during of which extreme movements occurred. Another interesting feature is the negative skewness of both stock returns, which means that the left tail of distribution is turn out to be longer. Least but not the last argument against non-normality is the phenomenon of highly excess kurtosis in all cases. For instance, the kurtosis of 10 and 8 for large and small caps respectively is significantly higher than the reference point of 3 that belongs to Normal distribution. Overall, one falls into doubt when observes such abnormalities in the data. Thus, our next steps are about getting stronger evidence in favour of tail heaviness and then to quantify it.

4. Methodology

4.1 Modelling Asset Returns With GARCH

From the statistical point of view different types of financial time series share some common statistical properties such as the absence of autocorrelations in return, or in other words it means that returns follow a ‘random walk’. Nevertheless, it does not implicate the independence of increments because some non-linear functions of returns exhibit volatility clustering (Cont, 2001). For that reason, we model our log-return using GARCH (1;1) with one period lag structure, which is sufficient for financial time series modelling according to Brooks (2008). GARCH was introduced by Engle (1982) and extended by Bollelrslev (1986). The extension was that the conditional past variance now affects the conditional current variance (2).

$$y_t = \mu + \varphi y_{t-1} + u_t, u_t \sim N(0; \sigma_t^2) \tag{1}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta \sigma_{t-1}^2 \tag{2}$$

Accordingly, these equations (1) and (2) are referred to as mean and variance specifications of GARCH (1;1). In the second equation the variance depends on its past and α_1 and β are non-negative constants. In detail, α_1 determines the short-run impact of u_{t-1}^2 on conditional variance and β is the long-run impact of past variance on current conditional variance. Afterwards, the software estimates parameters using maximum likelihood and the log likelihood function is given by equation (3).

$$L = -\frac{T}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\log(\sigma_t^2) - \frac{1}{2}\sum_{t=1}^{T}(y_t - \mu - \varphi y_{t-1})^2/\sigma_t^2. \tag{3}$$

By using the fact in equation (4),

$$u_t = v_t \sigma_t, v_t \sim N(0; 1) \tag{4}$$

we want to check the conditional normality assumption of standardized residuals (5),

$$v_t = \frac{u_t}{\sigma_t} \tag{5}$$

‘which would be the model disturbance at each point in time t divided by the conditional standard deviation at that point in time. Thus, it is the v_t that are assumed to be normally distributed, not u_t’ (Brooks 2008, p.399). Thus, the sample counterpart $\hat{v}_t$ can be tested using normality tests such as Jarque-Bera, by looking at quantile-quantile (Q-Q) plots and by comparing the kurtosis with standard normal kurtosis.

4.2 Tail Index Estimation

In order to model the frequency of extreme events in the tails, we apply the extreme value theory (EVT) techniques. EVT provides two different approaches, the block maxima (BMM) and peak over threshold (POT). They differ in a sense that POT practices more efficient usage of data, or in other words, BMM models the largest values from the data, while POT models only extreme observations based on the exceedance over a specific threshold (Embrechts et al., 1999). In our case, we apply the peaks over threshold approach, which follows so called generalized Pareto distribution (GPD). The cumulative distribution function of GPD is given by equation (6),

$$G_{\alpha,\beta}(x) = \begin{cases} 1-\left(1+\frac{\alpha x}{\beta}\right)^{-\frac{1}{\alpha}}, & \alpha \neq 0 \\ 1-\exp\left(1-\frac{x}{\beta}\right), & \alpha = 0 \end{cases} \quad (6)$$

where α and β are known as scale and shape parameters accordingly. In general, GPD is a probability distribution that models the left or right tail of another distribution without looking at the body of that distribution. GDP comes with two main issues. The first one is a need for a sufficient number of observations in the tail, which means that one needs a large underlying sample and the second the determination of the beginning and the end of the tail. Our goal is to estimate the tail shape parameter (Note 4) α that determines the "fatness of the tail", so that the lower the alpha the thicker the tails. Literature differentiates the tail index into three categories. Specifically, $\alpha < 0$, $\alpha = 0$ and $\alpha > 0$ are the cases of short, light and heavy tailed distributions respectively. And a random variable X has a heavy tail if 'the sample comes from a univariate α-stable distribution and observations asymptotically have a Pareto tail' (7),

$$P(X_1 > x) \sim x^{-\alpha} L(x), \quad x \to \infty \quad (7)$$

where α is tail index and L is slowly varying function (Rachev, 2003). The tail index represents the decay of tail to zero and there are three cases of α within the heavy tailed category (Finkenstadt and Rootzén, 2003). First, for $0 < \alpha < 1$ the distribution has very heavy tails with infinite mean and variance. Second, for $1 < \alpha < 2$ the distribution with heavy tails has only first finite moment. Finally, $\alpha > 2$ is commonly observed in financial time series and belongs to the case of distribution with finite variance. Another interpretation of tail index is that it shows the number of finite moments of the variable under consideration. For instance, if $\alpha > 4$, then variable has finite kurtosis, or when $\alpha > 2$ the variances are finite. According to recent studies, most of the financial returns have finite first and second moments but infinite kurtosis (Ibragivom, 2009; Gabaix et al. 2009). In other words, the existence of two moments is equivalent to stating that the distribution has finite mean and variance, which is essential for the soundness of many economic models, econometric and statistical approaches, such as value at risk analysis of profit and loss for investments and least square regression methods for the various economic and financial variables (Ibragimov and Walden, 2007). We employ Hill's (1975) popular tail index estimator, which is based on iid observations that is decreasingly ordered. Particularly, $X_{(1)} > \cdots > X_{(n)}$ is considered as the order statistics of our random sample. Then Hill's (1975) estimator is based on $k+1$ order statistics and given by equation (8),

$$\hat{\alpha}_{Hill} = \left(\frac{1}{k}\sum_{j=1}^{k} log X_{n+1-j} - log X_{n-k}\right)^{-1} \quad (8)$$

where k is the number of upper order statistics and $\hat{\alpha}_{Hill}$ is Hill estimator, which is proven to be consistent and asymptotically normal by Mason (1982) and Hall (1982) respectively. It is important to mention that the only problem with this estimator is an optimal choice of k and the well-known solution is given by "Hill Plot", which plots tail indexes for various levels of upper order statistics and then shape parameter is detected from the stable region of the graph. In detail, Hill plot is a function of $((k, \hat{\alpha}_{k,n}), 1 \leq k < n)$. In cases when the plot is volatile, Resnic and Starica (1997) suggest averaging or smoothing the 'Hill estimator' (avHill) values or to rescale the axis of k (threshold), so that small number of order statistics (k) is shown more clearly on the graph. The smoothed "Hill plot" averages different "Hill estimators" according to the number of order statistics (9),

$$av\hat{\alpha}_{k,n} = \frac{1}{(u-1)k}\sum_{p=k+1}^{uk}\hat{\alpha}_{p,n} \tag{9}$$

where $u > 1$ is smoothing factor. Technically this modification reduces the volatility of the plot by averaging and decreasing the variance of the estimator. For the same reason, they also recommend transforming k axis into logarithmic scale, which is referred to as alternative Hill plot (altHill), which plots $((\theta, \hat{\alpha}_{n^{\theta},n}), 0 \leq \theta \leq 1)$, where θ is a logarithmic scale of k.

4.3 Structural Breaks in Tail Behaviour

In order to find whether tail index changed during stock market crashes, we split our sample into two sub-samples and compare the estimated tail indices. The general method of comparison is done by testing the equality of those tail estimates by using their asymptotic normal distribution. We start with theorem 2 in Hall (1982), $k^{\frac{1}{2}}(\hat{\alpha}_k - a) \xrightarrow{d} N(0, a^2)$, which gives the asymptotic normal distribution of "Hill estimator" and which can be used to test whether the tail index is constant across sub-periods. The null and alternative hypotheses are,

$$H_0: a^{(1)} = a^{(2)} = a \qquad and \qquad H_1: a^{(1)} \neq a^{(2)} \tag{10}$$

However, Loretan and Phillips (1994) construct a test statistic for an equal sample split $n_1 = n_2 = n/2$ that is based on differences (11),

$$V_0 = \frac{k_{\{1\}}\left(\widehat{\alpha_{\{1\}}} - \widehat{\alpha_{\{2\}}}\right)^2}{\left(\widehat{\alpha_{\{1\}}} + \widehat{\alpha_{\{2\}}}\right)} \xrightarrow{d} \chi_1^2 \quad , \tag{11}$$

which follows a chi-square distribution. However, since we have unequal sample split, we refer to Quintos et al. (2001) who extends the above test for an unequal sample size (12),

$$V_1 = \frac{k_{\{1\}}\,\hat{\alpha}_{\{2\}}^2\left(\frac{\hat{\alpha}_{\{1\}}}{\hat{\alpha}_{\{2\}}} - 1\right)^2}{\hat{\alpha}_{\{1\}}^2 + \left(\frac{k_{\{1\}}}{k_{\{2\}}}\right)\hat{\alpha}_{\{2\}}^2} \xrightarrow{d} \chi_1^2 \tag{12}$$

We apply the extended version of the test to our tail behaviour analysis over time.

5. Results

5.1 Discussion of GARCH (1;1) Results and Diagnostics of Standardized Innovations

As pointed out in methodology part, we fitted GARCH (1;1) to our stock indices and Table 2 shows the outcomes that apparently satisfy the general features of these processes. Mainly, coefficients of ARCH and GARCH specifications sum up to one, which indicates the persistence of volatility (Chan, 2011). In addition, the coefficients of lagged value of residual (Note 5) and the past variance are statistically significant. Overall, the results confirm the idea of Brooks (2008) that GARCH (1;1) model is sufficient to capture the volatility clustering in the financial data. However, this type of modelling of returns assume the normal distribution of innovations, which has the same shape as the conditional distribution of future returns, so that any deviation from the normality assumption might have serious consequences such as undervaluation and overvaluation in financial risk management or in terms of option pricings (Sun and Zhou, 2014). Because of this and according to the literature, it is in our best interest to inspect the conditional distribution of future returns for the presence of fat tails. In the first place, inappropriateness of normal distribution to model returns can be shown by Q-Q plots. This method plots the quantiles of two distributions on vertical and horizontal axis, and difference in quantiles could be seen as a departure from the straight line that corresponds to standard normal distribution. Hence in Figure 1 the sample curves significantly deviate from the straight line both for small and large cap companies. In other words, quantiles of data does not match the quantiles of normal distribution in the upper and lower parts of the data, which implies the presence of heavy tails.

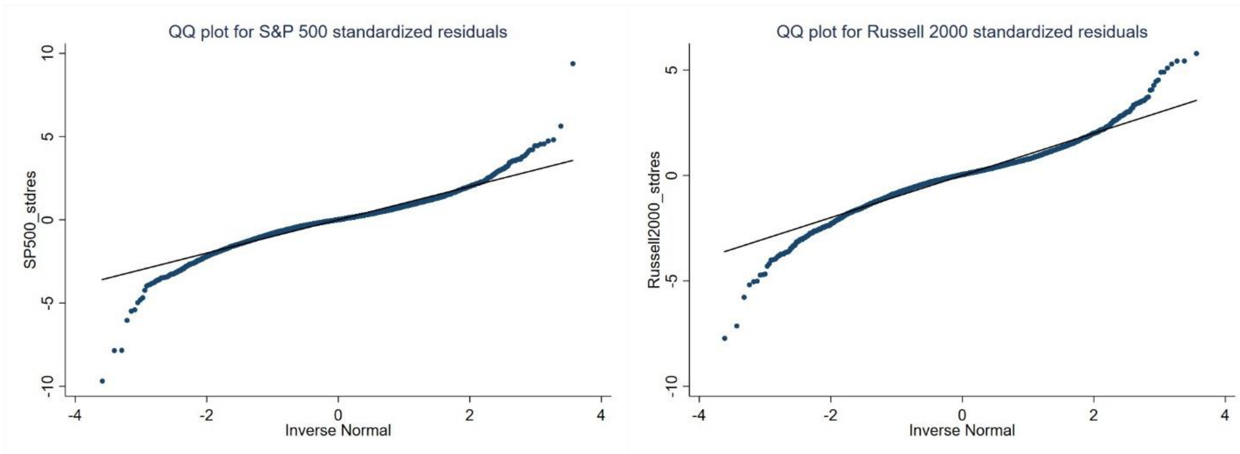

Figure 1

It can also be confirmed by the histograms of standardized GARCH residuals. Particularly, in both cases empirical distribution is symmetric, but there is significant amount of mass around the origin and in the tails compared to standards normal distribution (Figure 2).

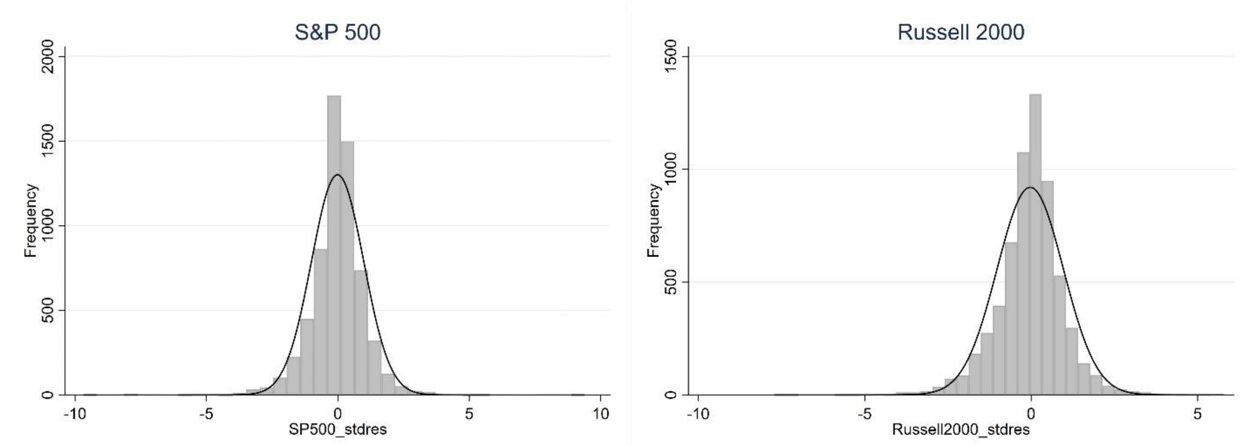

Figure 2. Histograms of standardized GARCH residuals (Note 6)

In spite of the fact that GARCH explains most of the non-linearity in our data, it can't explain the excess kurtosis, which is equal to 9.28 and 6.77 for large and small companies respectively (Table 3). Kurtosis higher than three (Note 7) is a strong evidence of leptokurtic distribution compared to standard normal distribution. Moreover, we also implement the test for normality assumption, which matches the 3rd and 4th moments with normal distribution, and reject the null hypothesis of normality (Table 4).

5.2 Tail Index Estimation

One of the prominent methods of estimating the tail shape parameter (α) is the "Hill estimator", which requires the pre choice of the number of upper order statistics (k). To overcome this issue we construct a "Hill plot", which graphs the different values of $\hat{\alpha}$ against the k, and then one can infer the value of the tail index from the stable region of the graph. Besides, being motivated by another stylized fact about financial returns, mainly the gain and loss asymmetry (Note 8), we analyze left and right tails of our estimated GARCH residuals separately, since they tend to experience unequal up and down movements (Cont, 2001). Moreover, we also construct various versions of classical Hill plots for the sake of comparison. These include avHill, altHill, smooHill and altsmooHill as discussed in methodology part. The subsequent two sub-sections discuss the tail index estimation for the cases of S&P 500 and

Russell 2000 accordingly. The software R studio automatically infers an estimate from the stabilised region of the graph. However, we still check for their validity using our best judgement. Table 5 summarizes the tail estimates inferred from different types of Hill plots and gives their standard errors and confidence intervals. In general, our sample consists more than 3000 observations per each tail, which is more than recommended by Goldberg et al. (2008)

Table 5. Estimated tail indices for S&P 500 and Russell 2000 (full sample)

		SP500		*Russell 2000*	
		Left	*Right*	*Left*	*Right*
Hill	*tail index* $\hat{\alpha}$	*3*	*2.9*	*3.1*	*2.9*
	standard error	*0.0192*	*0.0196*	*0.0188*	*0.0191*
	95% ci	*(2.7;3.3)*	*(2.8;3.7)*	*(2.8;3.5)*	*(2.6;3.3)*
	threshold k (Note 9)	*306*	*314*	*295*	*320*
	value of threshold (Note 10)	*1.6*	*1.5*	*1.7*	*1.4*
	% of sample	*9.9*	*9.8*	*9.7*	*9.8*
altHill	*tail index* $\hat{\alpha}$	*3*	*2.9*	*3.1*	*2.9*
	standard error	*0.0192*	*0.0196*	*0.0188*	*0.0191*
	95% ci	*(2.7;3.4)*	*(2.8;3.7)*	*(2.8;3.5)*	*(2.6;3.3)*
	threshold k	*306*	*314*	*295*	*320*
	value of threshold	*1.6*	*1.5*	*1.7*	*1.4*
	% of sample	*9.9*	*9.8*	*9.7*	*9.8*
smooHill	*tail index* $\hat{\alpha}$	*2.7*	*2.6*	*2.8*	*2.7*
	standard error	*0.0166*	*0.0168*	*0.0162*	*0.0164*
	95% ci	*(2.5;2.96)*	*(2.5;2.93)*	*(2.6;3.1)*	*(2.5;2.9)*
	threshold k	*306*	*307*	*295*	*322*
	value of threshold	*1.6*	*1.5*	*1.7*	*1.4*
	% of sample	*9.9*	*9.6*	*9.7*	*9.9*
altsmooHill	*tail index* $\hat{\alpha}$	*2.7*	*2.6*	*2.8*	*2.7*
	standard error	*0.0166*	*0.0168*	*0.0162*	*0.0164*
	95% ci	*(2.5;2.96)*	*(2.5;2.93)*	*(2.6;3.1)*	*(2.5;2.9)*
	threshold k	*306*	*307*	*295*	*322*
	value of threshold	*1.6*	*1.5*	*1.7*	*1.4*
	% of sample	*9.9*	*9.6*	*9.7*	*9.9*
	N	*3102*	*3189*	*3038*	*3252*

5.2.1 The Case of S&P 500

As planned, the classical Hill plots (Note 11) for S&P 500 are depicted in the Figure 3. On the face of it, the plots for left and right tails appear to be rather vague and volatile. In the literature they are called “Horror plots”, and as one of the solutions is to get a better picture by zooming the original Hill Plots in the 10% of the sample. Here, the zooming does not imply restricting the k axis by a fixed fraction of the sample (10%, 5% and etc). As expected, enlarged versions of plots allow us to extract the approximate value of shape parameter, since it seems to be stabilized in the neighbourhood of 3 for both tails (Figure 4).

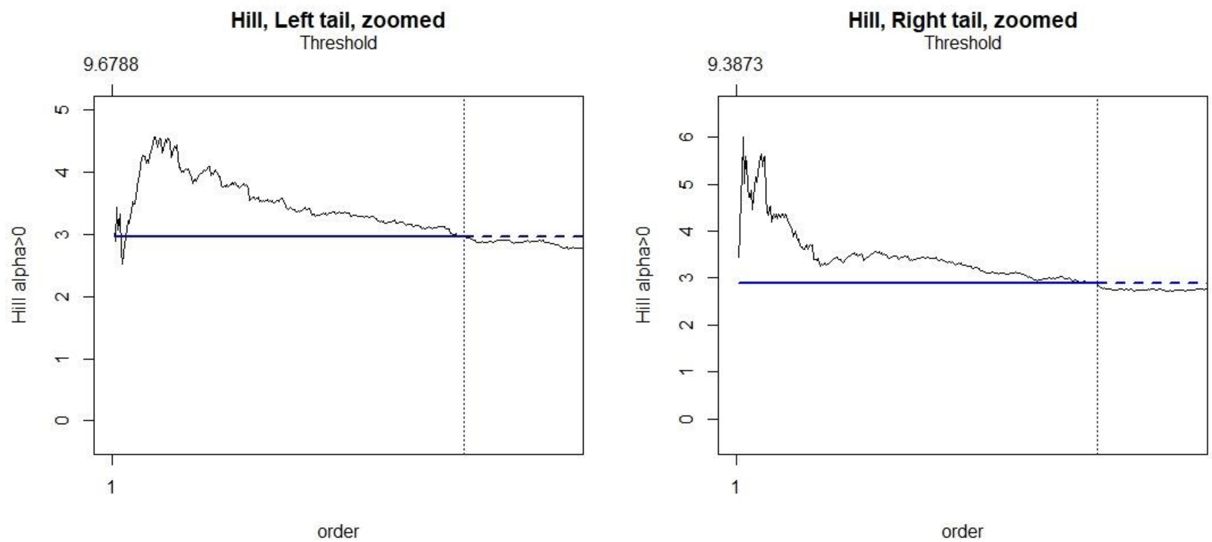

Figure 4. Zoomed Classical Hill Plots for S&P 500

This implies that the heavy left tail of our standardized GARCH residuals starts from corresponding threshold value of 1.6 and goes up to its largest value, or in other words it means that Hill estimator utilizes highest 306th observations (around 10%) from the sample in order to estimate the tail index (Table 5). This threshold of standardized residuals corresponds to a negative movement in terms of asset returns. The reason why it appears to be positive is that their absolute values were used as required by the "Hill estimator". In contrast, the right tail gives almost the same result in terms of tail index, threshold and confidence interval. Now we shift to the alternative Hill plot (AltHill) proposed by Dree et al. (2000). In this new method, a logarithmic scale for the axis of upper order statistics is used. Practically, it implies stretching the left side of the plot that gives a room for smaller observations and better picture, which is displayed in Figure 5. At first glance, we indeed observe the improvement over the visibility of elements to the left side of the graph, but it still remains uninformative. In spite of this fact, we still report the estimates given by the software for the sake of comparison. More interestingly, we could not apply the zooming technique in this particular case, because of the logarithmic scale is used on k axis. The next solution is to apply the averaging technique (smooHill (Note 12)) build by Resnic and Starica (1997) with respect to the various number of upper order statistics. This smoothing method decreases the variance of Hill estimator and thus it removes volatility of the plot to some extent. Accordingly, Figure 6 represents smoothed versions of Hill plots, while Figure 7 gives zoomed versions of these figures. It is apparent that the volatility of plots is completely erased, and looking at zoomed versions one can now easily confirm the value of estimated tail indexes that are $\widehat{\alpha_L} = 2.7$ for the left and $\widehat{\alpha_R} = 2.6$ for the right tails.

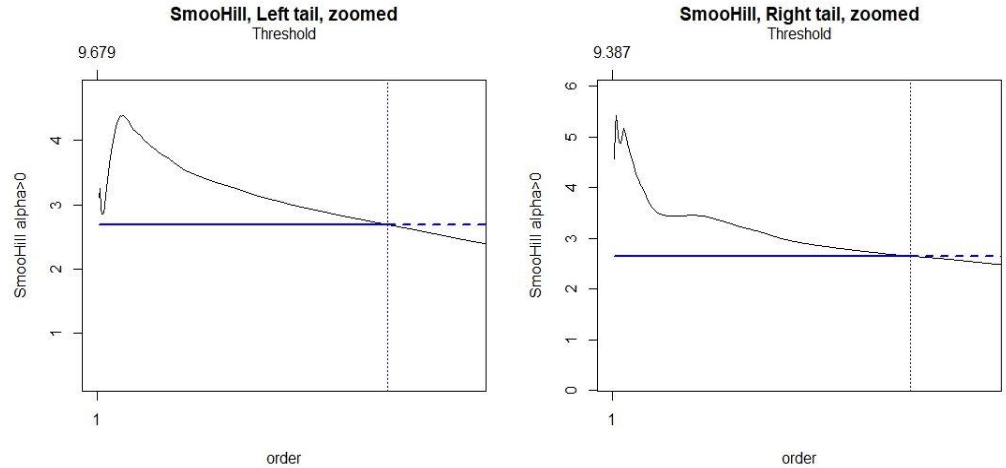

Figure 7. Smoothed Hill plots for S&P 500, zoomed version

Finally, we apply another advice by Drees et al. (2000), so called alternative SmooHill that is displayed on the Figure 8. In spite of the fact that altsmooHill erases the volatility of the plot, one could not see the stabilized region with naked eyes. With an aim of comparing methods between each other, we can refer to Table 5. Overall, we observe an improvement in terms of tail estimates when we use smooHill and altsmoHill. This is in a line with our expectation in a sense that (Drees et al., 2000) has shown its superiority over other versions of Hill plots. The second interesting remark is that left and right tails does not differ in terms of tail estimates, so that we do not observe any distributional imbalances. This however contradicts to the stylized fact of asymmetry of asset returns. More interestingly, the threshold number of observations amounts for 10% of the sample for both tails across all types of Hill plots, which confirms the popular suggestion of DuMouchel (1982) about the 10% fixed fraction selection rule for threshold.

5.2.2 The Case of Russell 2000

Turning to the analysis of another composite index-Russell 2000, we again observe the vagueness of classical Hill Plots in Figure 9. The same procedure in the case of S&P 500 is applied here. First, we look at the zoomed versions at 10% of the sample, which clearly shows the stabilization area around the 3.1 and 2.9 for left and right tails respectively (Figure 10).

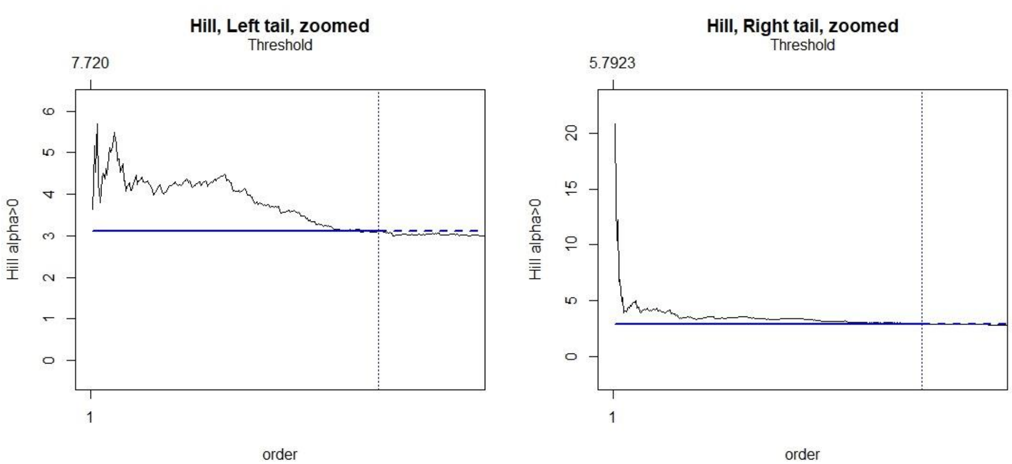

Figure 10. Zoomed Classical Hill Plots for Russell 2000

Then we shift to altHill plots, where in the right tail one can clearly see the stabilized area and infer an estimate of 2.9 (Figure 11). However, there might be some doubts about the left tail's estimate of 3.1 by software, because it does not seem to stabilize in that region. Therefore, we apply the smoothed version of Hill plot and get an estimate of 2.8 and 2.7 for each tail respectively (Figure 12). However, we think that plots are not informative and thus try to look at the zoomed versions in order to get better picture. And indeed, Figure 13 illustrate the clearly stabilized are at $\hat{\alpha}_R = 2.7$ on the right, but not so really convincing $\hat{\alpha}_L = 2.8$ on the left.

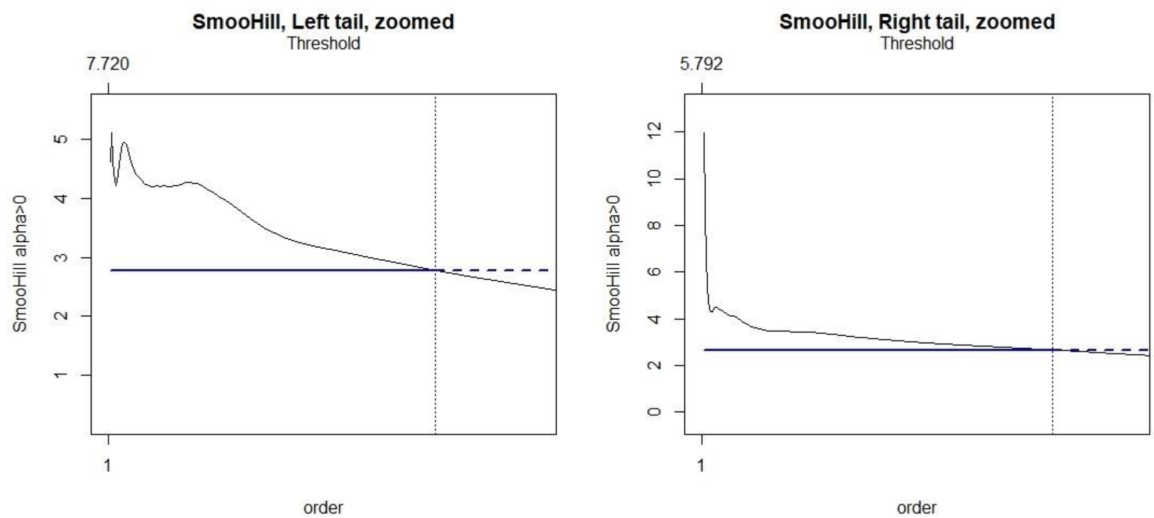

Figure 13. Zoomed smooHill Russell 2000

And we finish the "threshold finding" process by constructing the alternative smoothed Hill plots (Figure 14). Obviously, we could not say anything about these unstable graphs, but still report the estimates provided by software in the Table 5. Overall, for the case of Russell 2000, we observe different tail estimates across different types of Hill plots, which indicates the superiority of some versions over others, which are confirmed by our investigation. As in the case of S&P 500, the tails turn out to be similar, since they have almost the same shape and threshold levels, which is again contradicts to the expectation of the theory. Interestingly, every tail of large caps includes approximately 10% of the sample in the tail estimation process, which is in line with DuMouchel's (1982) rule of thumb.

5.2.3 Main Findings and Inference on the Shape Parameter

The evidence acquired from the diagnostic analysis for the presence of heavy tails indicates that both left and right conditional tails of S&P 500 and Russell 2000 accordingly lie in the domain of attraction of Generalized Pareto distributions (GPD). Overall, our results for tail index estimates lie in the interval $\hat{\alpha} \in [2.6; 3.1]$ for all types of "Hill plots". Thus, we assume that the data does not come from stable distribution and has finite first and second moments. We reject the existence of the fourth moment at 95% confidence level in all cases. These results are mostly observed in financial time series (Gabaix, 2009; Ibragimov, 2009). Results also show that the kurtosis fails to converge in the limit, which is in line with findings of Loretan and Phillips (1994). The infiniteness of fourth moments comes with severe consequences, such as working with techniques based on autocorrelation functions (Granger and Orr, 1972).

In the interest of comparison of tail indices of two stocks, we stick to Smooth Hill estimates, since they are proven to be superior to standard methods (Drees et al., 2000) and shown to be valid in the above discussions. Therefore, left tail index $\widehat{\alpha_L} = 2.7$ for the standardized GARCH residuals of S&P 500 returns is lower than the corresponding tail index $\widehat{\alpha_L} = 2.8$ for the case of Russell 2000. The situation is the same for winner stocks, where $\widehat{\alpha_R} = 2.6$ and $\widehat{\alpha_R} = 2.7$ for large and small companies. Based on this we can conclude that both tails of S&P 500 are a little bit heavier than the tails of Russell 2000. It implies that there is a high tendency for extreme changes in stock returns for larger companies, which is in conformity with the conclusion of (Cenezoglu, 2008), who states that portfolios of large companies react stronger than the small companies to the macroeconomic news over time, such as interest rates, employment rate and etc. In addition, the tail estimates for left and right tails are approximately equal to each other with similar threshold level, which is against the hypothesis of profit and loss asymmetry.

5.3 Structural Breaks in the Tail Behavior

The significance of the structural breaks during the times of market turbulences in the tails is pointed out above. In 2008, humanity experienced one of the serious global financial crises in its history since the Great Depression. Therefore, we estimate the tail index for two sub-periods and look whether it changes or not. For instance, a fall in tail index would imply an increase in the probability of extremal events. Sub-periods include 2008 pre- and post-crash years. The summary of tail indices for each sub-sample is given in Table 6. Merely looking at tail indexes for the pre- and post-crash sample one notices that all tails became heavier after the global financial crisis. In order to test for the tail behavior over time we employ the test statistics provided by Quintos et al. (2001) (Note 13). Evidence shows that we reject the null hypothesis of constant tail index at 5% critical level both for S&P 500 and Russell 2000 composite indices and for their respective left and right tails (Table 7).

5.3.1 Implications of Changes in Tail Indices

Results imply the existence of so called “volatility regimes” where markets are represented by persistently low/high phases with different market characteristics during the market cycles in contrast to the average volatility (Peters, 2009). In other words, the fact that the tails become heavier after a certain breakpoint means that one should expect large up/down movements in the markets. This tells us that the standard volatility measures can’t alone explain the behavior of tails and assessing market risk based solely on those measures might be erroneous and have serious consequences in terms of adequately prevents those “tails risks” (Werner and Upper, 2004). Thus, after 2008 crisis there has been an increased awareness towards hedging the “black swan” events.

5.3.2 Questioning the Number of Breakpoints

We are curious about the number of breakpoints in tail indices in our full sample. Preliminarily, we look at the behaviour of standardized log return across time. Since our sample ranges from 09/09/1987 to 07/18/2019, we divide it into 4 equal sub-samples. We do it since we have a suspicion that there were other structural breaks in the tails of condition asset returns. The first sub-sample spans from 1987 to 1994, which includes the 1987 “Black Monday” stock crash in US. The second sample goes to the “Asian financial crisis” that lies in the range from 1995 to 2003. Then, the period of 2004-2012 contains the 2008 Global financial crisis. Finally, the time period 2013-2019 includes the European Debt crisis. De-facto, we expect the tails to become heavier after the main events. Figure 15 and Figure 16 graphically illustrates the behaviour of tail indices for large and small caps respectively throughout the newly created time spans. For the case of S&P 500, we observe that both right and left tails move in the same direction, becoming heavier only during the period that includes 2008 crisis. Interestingly, the tail indices rise during the Asian crisis, which might indicate the severity of 1987 stock market crash and consequent recovery. Shifting to Russel 2000, we observe the same picture as in the previous case. Based on this informal investigation one can conclude that there was only one structural break in our sample.

6. Conclusion

This paper has discussed the heavy tailed nature of conditional asset returns of large and small companies. The first question of our interest was whether our modelled asset returns present any signs of tail heaviness. This has been addressed by fitting log returns with GARCH (1;1) process, which give sound results. In the next step, after utilizing different diagnostic analysis of standardized GARCH residuals we find sufficient evidence for the presence of tail heaviness, so that the conditional empirical distribution of our log returns does not follow standard normal distribution. Mainly this has been done as follows. First, we compare the distribution of our sample with standard normal distribution using Q-Q plots. In addition, the same comparison is done by constructing histograms, where we observe some peculiarities, implying that there are too many observations peaked around the mean and that the distribution of returns has long tails. We also employ the Jarque Bera test and reject the null hypothesis of normality and encounter with excess kurtosis that is way above the level of normal density. The next logical step is to calculate the tail index, which summarizes the tail behavior. In order to do this, the traditional “Hil estimator” is used for the conditional asset returns. In this spirit, we overcome the problem of choice of threshold by plotting the values of “Hill estimator” on the vertical and the number of upper order statistics on the horizontal lines and then we look for the stabilized region of the plot to find the estimated tail index. Hill plots might be uninformative when the plot is volatile. We follow the recommendations of Deed et al. (2001) to decrease the variance of “Hill estimators” by simply averaging them, or as mentioned in Resnic and Starica (1997) to rescale the k axis and stretch the plot by giving more room for smaller observations. Initially, the classical Hill plots for all stocks appear to be uninformative, and the application of modified Hill plots help us to acquire the tail index. In general, when it comes to inference on the estimated tail indices, we rely on the so called smooHill estimates since we practically confirm their superiority among other methods as mentioned in the Deed et al. (2001). According to this, the tail estimates obtained by averaging technique always less than three and more than two $(2 < \hat{\alpha} < 3)$ according to their confidence intervals, which implies that the conditional distribution of asset returns is not a stable one and has finite mean and finite variance. This is often observed in financial time series (Gabaix, 2009; Ibragimov, 2009). Another interesting finding is that the tails of large and small companies are different, which confirms the proposition of Cenezoglu (2008). Third, we do not observe any distributional imbalances in all cases. This is in odds with a hypothesis of gain and loss asymmetry (Cont, 2001). And the last but not the least finding is that the thresholds in the process “Hill estimation” amounts for 10% of the sample for both left and right tails of S&P 500 and Russell 2000 respectively. Interestingly, it confirms the advice of DuMouchel (1982) to choose the fixed fraction (10%,) of the sample in order to avoid overfitting of the tails. Overall, the presence of heavy tails is of huge importance for the risk management, Value-at-Risk (VaR) calculations and many other economic models.

The second issue were concerned with the tail invariance over time. Otherwise stated, can highly volatile periods like the 2008 Global Financial Crisis affect the tail characteristics of conditional distribution of returns under consideration? For instance, if they become heavier after a crisis, it is interpreted as an increase in the probability of positive and negative events. In order to find out whether conditional tails of asset return experience abrupt changes during the Global Financial crisis, we divide our sample into the pre 2008 and post 2008 sub samples. To test this, we refer to test statistics created by Loretan and Phillips (1994) and modified for the case of unequal sample sizes by Quintos et. al (2001). We reject the null hypothesis of tail invariance over time on 5% significance level. We conclude that the conditional tails of large and small caps experienced structural break during the 2008 Global Financial Crisis. Additionally, we informally check the presence of other structural breaks in the tails throughout our full sample. Thus, based on this verbal analysis we can state that there was only one date of change in the tails of conditional asset returns that occurred during the 2008 crisis.

In a nutshell, our findings might be catastrophic for financial risk management, forecasting, option pricing, Value-at-Risk (VaR) calculations and others in terms of inconsistent results. Thus, this paper encourages to reconsider the normality assumption of those economic models.

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Appendix

Tables

Table 1. Descriptive Statistics for the Composite Stock Returns

Returns	Obs	Mean	Std.Dev.	Min	Max	p1	p99	Skew.	Kurt.
S&P 500 full sample	6229	0	0.011	-0.095	0.102	-0.03	0.029	-0.259	10.099
Russell 2000 full sample	6229	0.001	0.012	-0.1	0.081	-0.034	0.034	-0.258	8.607
S&P 500 before crisis	3953	0	0.01	-0.07	0.056	-0.026	0.026	-0.134	6.647
S&P 500 after crisis	2276	0	0.012	-0.095	0.102	-0.035	0.035	-0.371	11.903
Russell 2000 before crisis	3953	0.001	0.01	-0.075	0.057	-0.028	0.027	-0.234	5.91
Russell 2000 after crisis	2276	0	0.015	-0.1	0.081	-0.044	0.046	-0.238	7.785

Table 2. The Results of Fitting GARCH(1;1) for Both Stocks and for Full Sample

SP500	Coef.	St.Err.	t-value	p-value	[95% Conf	Interval]	Sig
Constant	0	0	3.83	0	0	0.001	***
L.arch	0.263	0.016	16.09	0	0.231	0.294	***
L.garch	0.714	0.049	14.47	0	0.617	0.81	***
Constant	0	0	0.78	0.438	0	0	

Mean dependent var	0	SD dependent var	0.011
Number of obs	6291	Chi-square	.

Prob > chi2	.	Akaike crit. (AIC)	-39829.007

*** p<0.01, ** p<0.05, * p<0.1

Russell2000	Coef.	St.Err.	t-value	p-value	[95% Conf	Interval]	Sig
Constant	0.001	0	6.7	0	0.001	0.001	***
L.arch	0.336	0.017	19.77	0	0.303	0.369	***
L.garch	0.749	0.04	18.71	0	0.671	0.828	***
Constant	0	0	-2.8	0.005	0	0	***

Mean dependent var	0.001	SD dependent var	0.012
Number of obs	6290	Chi-square	.
Prob > chi2	.	Akaike crit. (AIC)	-38464.628

*** p<0.01, ** p<0.05, * p<0.1

Table 3. Descriptive Statistics

Variables	Obs	Mean	Std. Dev.	Min	Max	p1	p99	Skew.	Kurt.
SP500 stdres	6291	-0.013	0.994	-9.679	9.387	-2.801	2.561	-0.311	9.282
Russell2000 stdres	6290	-0.029	0.996	-7.72	5.792	-2.832	2.543	-0.312	6.767

Table 4. Test of Normality

Variable	Obs	Pr(Skewness)	Pr(Kurtosis)	Joint Prob>chi2
SP500 stdres	6291	0	0	0
Russell2000 stdres	6290	0	0	0

Table 6. The Summary of Tail Indices (Note 14) of S&P 500 and Russell 2000 for Sub-samples Based on SmooHill

	S&P 500					
		Left			**Right**	
	alpha	se	k	alpha	se	k
1988-2007	3	0.018581	200	2.8	0.019584	193
2008-2019	2.3	0.033132	106	2.3	0.032493	106

Russell 2000						
	Left			Right		
	alpha	se	k	alpha	se	k
1988-2007	3.1	0.018308	193	2.8	0.019584	193
2008-2019	2.5	0.030192	106	2.3	0.03275	103

Table 7. Structural Change Tests

	Test statistics		
	Left	Right	**Critical 5%**
S&P 500	8.415727	4.176269	3.841
Russell 2000	5.079763	4.087626	3.841

Figures

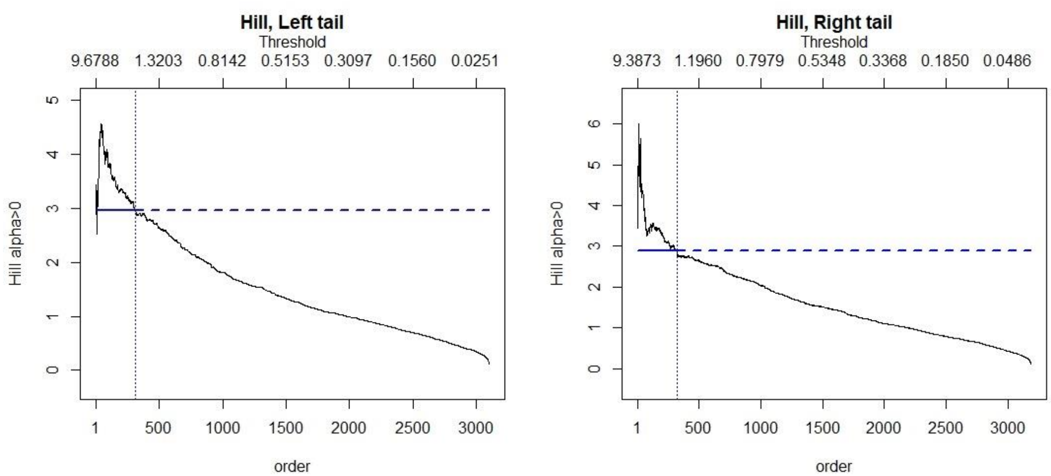

Figure 3. Classical Hill Plots for S&P 500

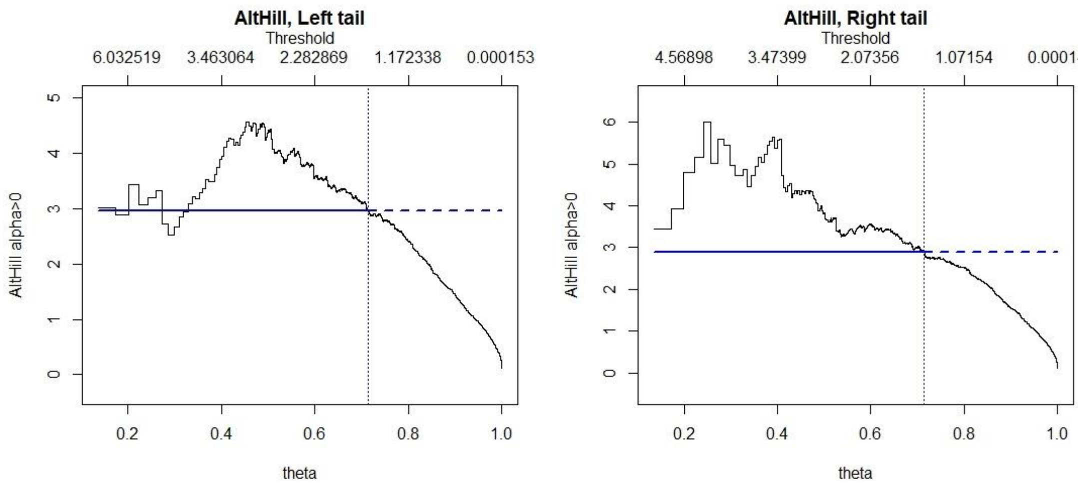

Figure 5. Alternative Hill Plots for S&P 500

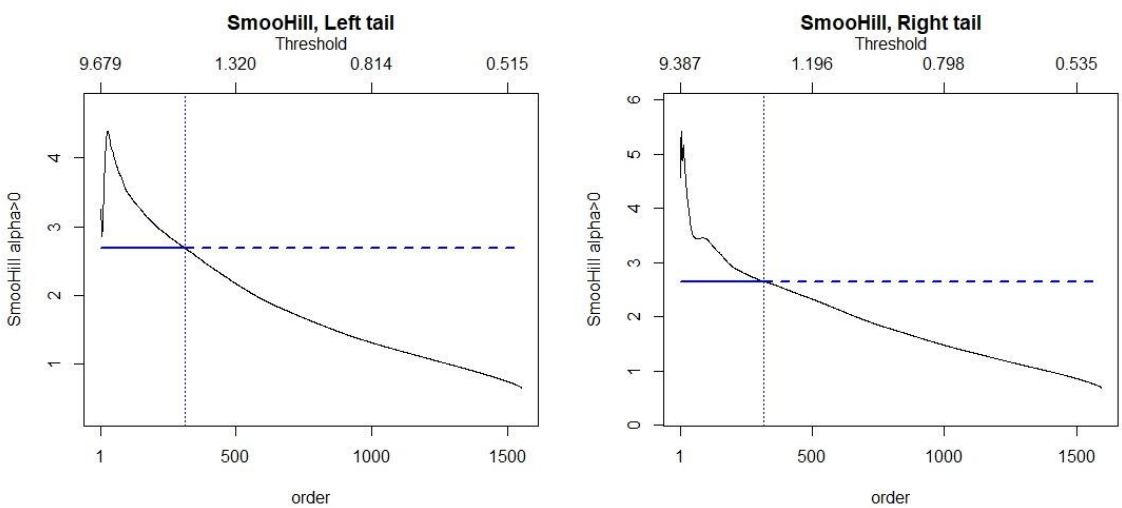

Figure 6. **Smoothed Hill Plots for S&P 500**

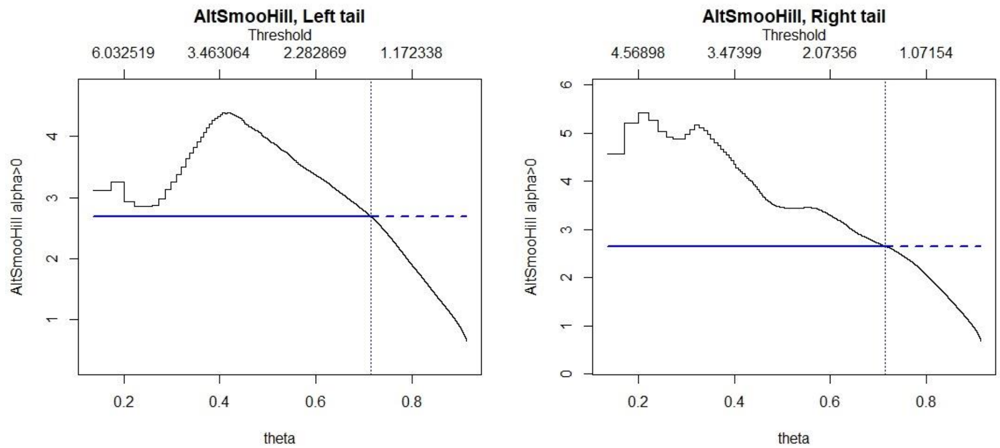

Figure 8. Alternative Smooth Hill Plots for S&P 500

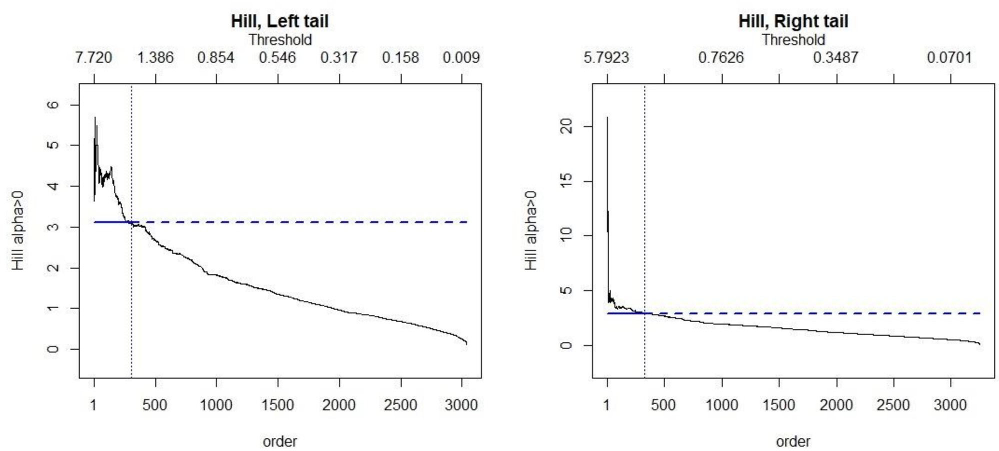

Figure 9. Classical Hill Plots for Russell 2000

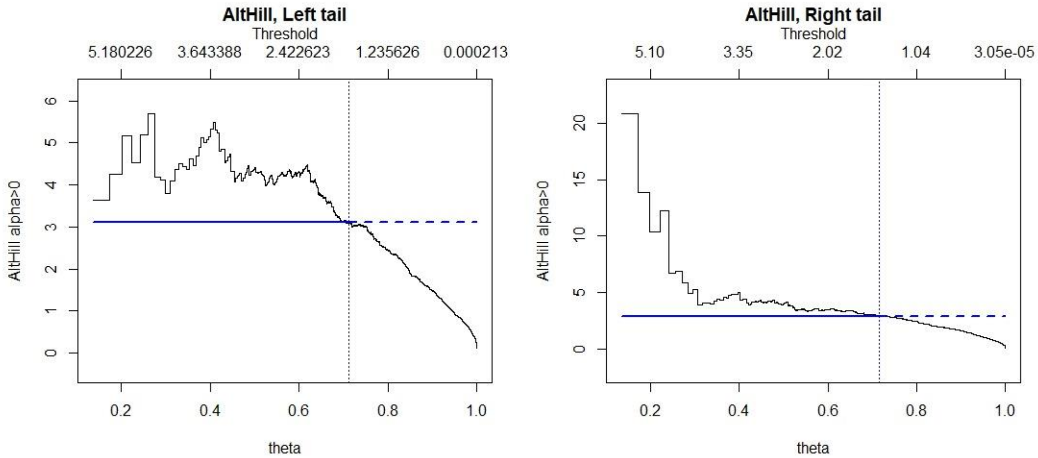

Figure 11. Alternative Hill Plots for Russell 2000

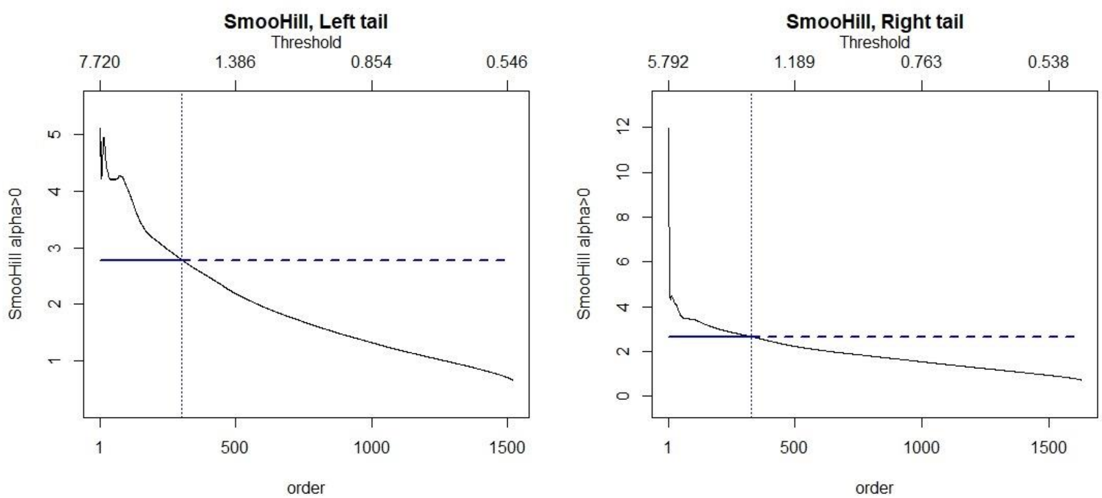

Figure 12. Smoohill Russell 2000

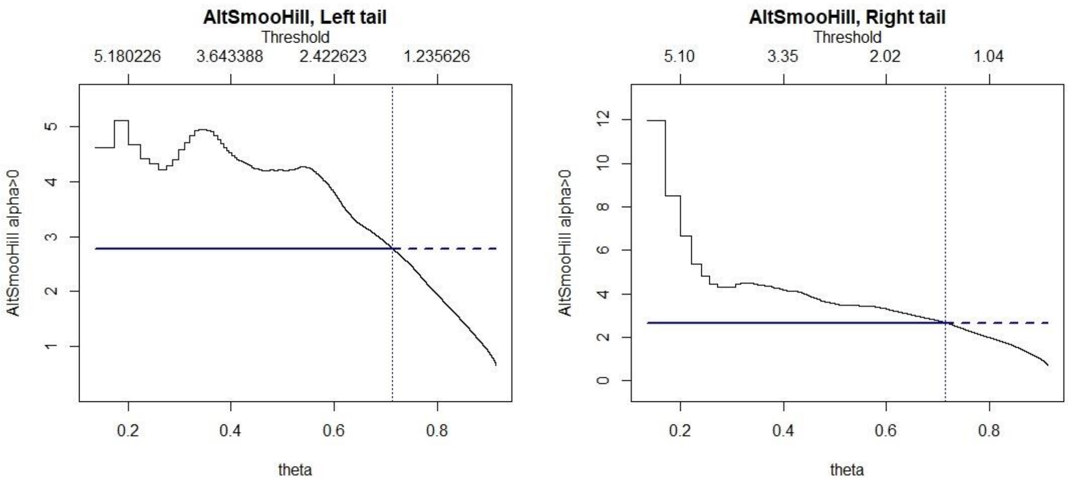

Figure 14. Alternative Smooth Hill Plots for Russel 2000

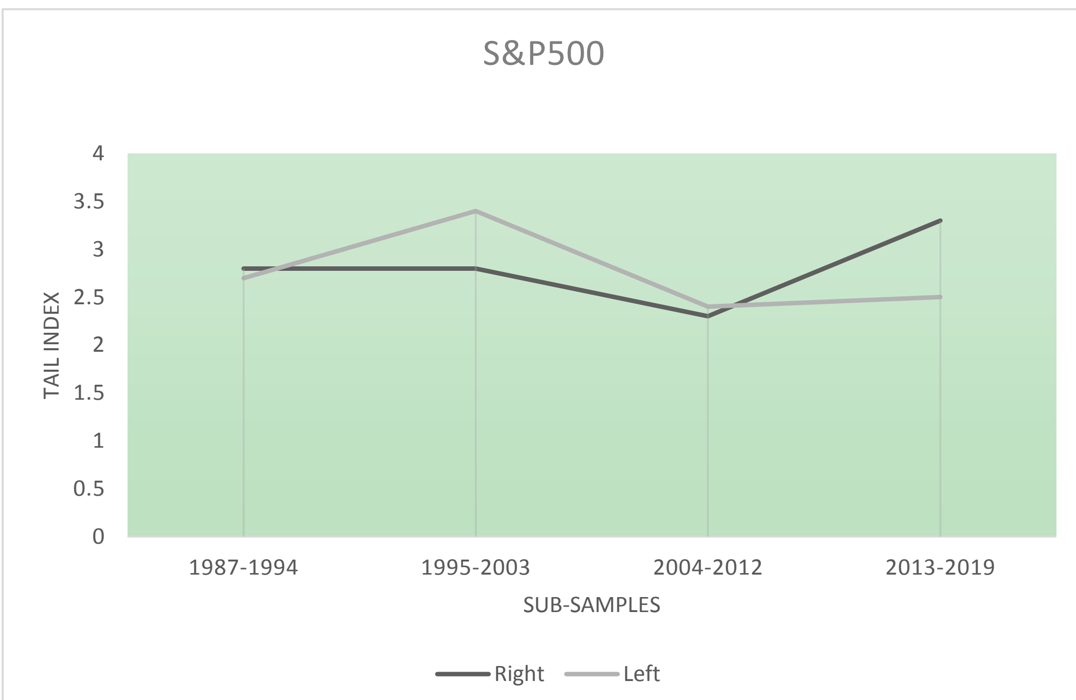

Figure 15

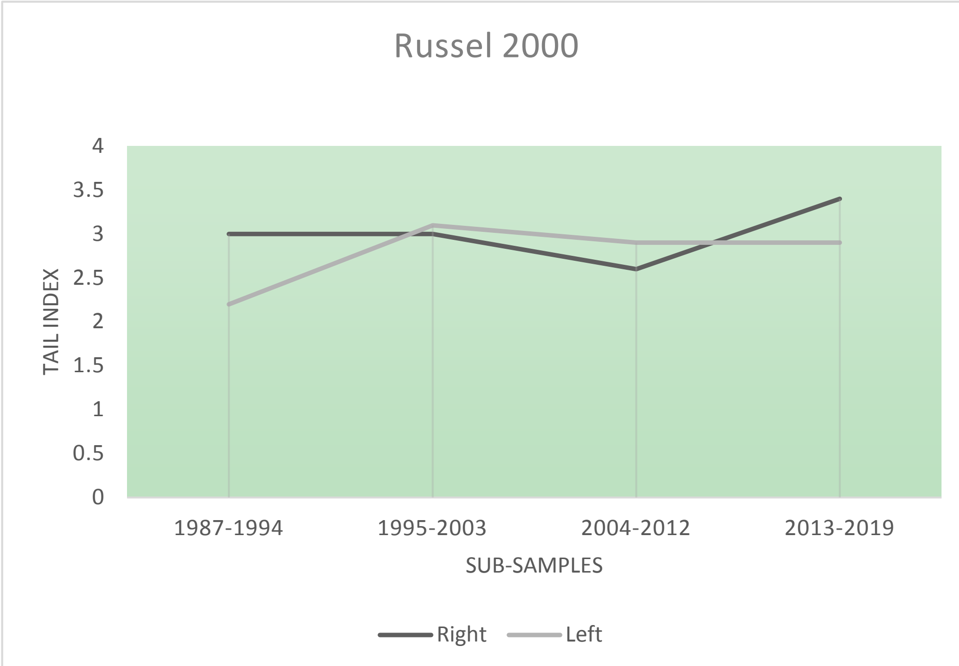

Figure 16

Notes

Note 1. Companies with relatively small market capitalization.

Note 2. We start our sub-sample from 1988 to avoid the 1987 "Black Monday" stock market crash.

Note 3. Both the largest loss of -0.095 and profit of 0.102 goes to S&P 500.

Note 4. Also known as "tail index".

Note 5. From the mean specification (see methodology section)

Note 6. The solid dark blue line represents the normal density.

Note 7. 3 corresponds to normal distribution.

Note 8. In theory also referred to as loss and profit; or loser and winner stocks.

Note 9. This is the number of upper order statistics. For example, 306 means that Hill estimator used highest 306 observations.

Note 10. The value of standardized GARCH residual corresponding to k^{th} upper order statistics.

Note 11. Software gives the estimated tail index from the stabilized area of the graph, which can be seen on the intersection of horizontal blue and vertical dashed lines.

Note 12. Smoothing factor is set to 2

Note 13. See the methodology.

Note 14. Only estimator based on SmooHill is given, since it is proven to be superior to others (Drees et al., 2000).

Note 15.