Counterparty Credit Risk on a Standard Swap in "Risky Closeout"

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Abstract

This paper provides risky closeout amount in computing counterparty credit risk. Under this closeout, we obtain a new nonlinear PDE model describing the value of a standard interest swap with counterparty credit risk in the reduced form framework, thus get a new method to calculate counterparty credit valuation adjustment. We solve the nonlinear PDE by iterations numerically and prove the convergence of this approach. By numerical examples, we show the difference between risky closeout and conventional closeout in estimating counterparty credit risk.

Keywords: Counterparty credit risk, Counterparty credit valuation adjustment, Interest swap, Nonlinear PDE.

1. Introduction

"The counterparty credit risk (CCR) is defined as the risk that the counterparty to a transaction could default before the final settlement of the transaction's cash flow. An economic loss would occur if the transactions or portfolio of the transactions with the counterparty has a positive economic value at the time of default."(Basel Π, AnnexIV, 2/A).

The sub-prime crisis has highlighted the importance of CCR in OTC derivative markets. This topic has already received a lot of attention in many papers published recently. To quote but a few: Brigo and Pallavicini [5] consider CCR with the reduced form approach and analyze the correlation between the hazard rate and interest rate. Brigo and Copponi [3] extend the work on the unilateral CCR. Leung and Kwok [15] take an intensity contagion model into account and provide the insight on how CCR influences the swap rate in a CDS. Lipton and Sepp [13] develop the structural approach with the value of the underlying asset with a jump diffusion process. Hui Li [11] focuses on a CDS with a stochastic recovery rate.

All papers mentioned above tried to describe the relation between wrong way risk and CCR or that between macroeconomic variables and CCR. However, a major issue, which is how to compute the closeout amount in estimation of CCR, is seldom mentioned (Note 1). The closeout amount is the net present value of the residual deal which is computed when one party defaults, and that is used for default settlement. We point out that most of existing literature assumes that, at the moment of default, a risk free closeout amount (note 2) will be used.

Here, we argue that a 'risky closeout amount' should be used, which means when a net present value of the residual deal is computed, the transaction should be considered as one with CCR. Different from a risk free one, the risky closeout in computation is more complicated, but more consistent with the definition of CCR. Thus we expect that the risky closeout will be more accurate in calculating counterparty credit valuation adjustment (CVA)—the measurement of CCR.

To support this statement, we look back to the definition of CCR, which implies when the counterparty defaults, there will be no effect on the investor if the value of the transaction to the investor's position is negative. It can be seen that the risky closeout coincides with this implication, since the investor will pay the total risky transaction's value to the counterparty in this situation. By contrast, in risk free closeout, when the counterparty defaults and the transaction has positive value from the counterparty's perspective, the investor will pay the risk free value of the transaction, rather than the fair value of the defaultable transaction, to the counterparty. Since the risk free value is greater than the fair value intuitively, the default makes profit for the counterparty. Therefore, the conventional risk free closeout amount is inconsistent with the definition.

In this paper, to show how to estimate CCR in risky closeout, we focus on a standard interest swap with CCR. Since the swap is interest sensitive, we will establish math model in reduced form approach which can relate interest rates to the credit event.

The paper is structured as follows. In section 2, we first describe the mechanism and cash flows of a standard interest swap with CCR under risky closeout. Also the appropriate notation is mentioned. We then derive a general formula for pricing the swap with CCR in risky closeout and thus obtain a formula for CVA. In section 3, under reduced form framework, we obtain a nonlinear PDE to compute the swap with CCR, thus get a new approach to estimate CVA. In section 4, to solve the nonlinear PDE, an iteration method is provided and the convergence of this approach is proved in appendix. In section 5, we give numerical examples, in which the difference between CVAs computed in risky closeout and risk free closeout is shown. In section 6, we conclude the paper.

2. General Set-up

2.1 Cash flows

A standard interest swap involves two entities: one party (the investor) paying interest at a floating rate, the other (the counterparty) at a fixed rate. The principal is notional in the sense that it is never paid by either party; it is merely used to determine the magnitudes of the payments. The issue of counterparty risk on a standard swap is:

- Primarily, the fact that the investor may fail to pay the floating rate
- Also the symmetric concern that the counterparty may fail to pay the fixed rate.

In this paper, we focus on 'unilateral CCR', namely, the risk corresponding to the second bullet point above.

Let us fix a period δ and a set of dates $T_j = j\delta,\ j = 0,1,2,\dots, N+1$, considering a defaultable standard swap with payment dates $T_1, T_2, \dots T_{N+1}$ on a notional principle 1. At each T_j, the investor receives $k\delta$, the simple interest accrued on a principal of 1 over interval of length δ at an annual rate of k, and the counterparty receives L_{j-1}, the simple annualized interest rate fixed at T_{j-1} for the interval $[T_{j-1}, T_j]$. The exchange of payments terminates at the counterparty's default time τ or the transaction's maturity $T\ (= T_{N+1})$, whichever comes first.

Let us denote by R the recovery rate of the counterparty, supposed to be constant in this paper, a fair value V_τ of the defaultable contract is computed at time τ. If this value (from the perspective of the investor) is positive, the counterparty is assumed to pay to the investor RV, whereas the value is negative, $-V_\tau$ is paid by the investor to the counterparty. Therefore, when the counterparty defaults before the maturity, the closeout amount is $RV^+ - V^-$.

Remark 2.1 The conventional closeout amount (risk free closeout amount) computed when the counterparty defaults before the maturity is $RU^+ - U^-$, where U is the value of a standard interest swap without CCR. We shall see in section4, the two kinds of closeouts make considerable difference in practice.

2.2 pricing

We introduce a filtered probability space $(\Omega, G, \{G_t\}_{t\geq 0}, P)$ to describe the uncertainty of the market. The filtration $\{G\}_{t\geq 0}$ represents the flow of information of the market. P is risk neutral measure on $G\ (= \vee G_t)$. τ is G_t -stopping time. We follow the usual assumption for G_t, namely $G_t = H_t \vee F_t$, where $H_t = \sigma\{I_{\tau\leq s}, s \leq t\}$ and $\{F_t\}_{t\geq 0}$ contains available market information which is generated by a certain (some) stochastic process (processes). All the cash flows and prices are considered from the perspective of the investor. In view of the description of the cash flows in subsection 2.1, we have

Definition 0 The value of a standard interest swap with CCR is given by.

$$V_{T_i} = E[\sum_{j=i+1}^{N+1} k\delta e^{-\int_{T_i}^{T_{N+1}} r_\theta d\theta} I_{\tau>T_j} \mid G_t] + E[I_{\tau>T_i}(e^{-\int_{T_i}^{\tau\wedge T_{N+1}} r_\theta d\theta} - 1) \mid G_t] + E[I_{T_i<\tau<T_{N+1}} e^{-\int_{T_i}^{\tau} r_\theta d\theta}(RV_\tau^+ - V_\tau^-) \mid G_t] \quad (2.1)$$

The first term is the value of fixed interest rates receiving by the investor. The second term corresponds to the value of floating rates paying by her. The final term measures the value of the risky closeout amount when the counterparty defaults before the maturity. If we replace $RV_\tau^+ - V_\tau^-$ by $RU_\tau^+ - U_\tau^-$, where U is the swap value without CCR, the risk free closeout is applied.

In the rest of the paper, for simplicity, we assume that the floating rates and fixed rates are paid continuously. We have the continuous form of the definition 0.

Definition 1 The value of the swap with CCR is.

$$V_t = E[\int_t^T ke^{-\int_t^s r_\theta d\theta} I_{\tau>s>t} ds \mid G_t] + E[I_{\tau>t}(e^{-\int_t^{\tau\wedge T} r_\theta d\theta} - 1) \mid G_t] + E[I_{t<\tau<T} e^{-\int_t^\tau r_\theta d\theta}(RV_\tau^+ - V_\tau^-) \mid G_t] \tag{2.2}$$

Let $p(\tau = \infty) = 1$, we have

Definition 2 The value of the swap without CCR is

$$U_t = E[\int_t^T ke^{-\int_t^s r_\theta d\theta} ds \mid G_t] + E[e^{-\int_t^T r_\theta d\theta} - 1 \mid G_t] \tag{2.3}$$

To compute the price of CCR, we have

Definition 3 As the price of counterparty credit risk, the counterparty credit valuation adjustment (CVA) is given by

$$CVA_t = U_t - V_t \tag{2.4}$$

3. Nonlinear PDE Model under Reduced Form Framework

3.1 Reduced form models

In reduced form models, the default event is specified in terms of an exogenous stochastic process—a hazard rate. The specification of the stochastic process is flexible so that the time of default can be related to some economic variables. Since a standard interest swap is sensitive to interest, reduced form models are used in this paper.

We assume that the hazard rate λ_t is satisfied with

$$P(\tau > t \mid F_t) = e^{-\int_0^t \lambda_\theta d\theta} \tag{3.1}$$

λ_t is nonnegative F_t progressively stochastic process.

Under the above assumption, we have

Theorem 3.1 The value of the swap with CCR under reduced form framework can be presented as

$$V_t = I_{\tau>t} E[\int_t^T ke^{-\int_t^s(\lambda_\theta+r_\theta)d\theta} ds + \int_t^T \lambda_s e^{-\int_t^s(r_\theta+\lambda_\theta)d\theta} ds + e^{-\int_t^T(\lambda_\theta+r_\theta)d\theta} - 1 + \int_t^T \lambda_s e^{-\int_t^s(r_\theta+\lambda_\theta)d\theta}(RV^+ - V^-)ds \mid F_t] \tag{3.2}$$

The proof is shown in appendix A.

3.2 Nonlinear PDE model

We denote economic variables which are correlated to the counterparty credit risk by

$X_t = (X_t^1, X_t^2, \dots, X_t^n)$, that is, $F_t = \sigma(X_s, s \le t)$. The process X_t is a diffusion process following the stochastic differential equation

$$dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dW_t,\ X_0 = x \in D. \tag{3.3}$$

With an m -dimensional Brownian motion $W_t \in R^m$ and function $\mu : [0,T] \times D \to R^n$,

$\sigma : [0,T] \times D \to R^n \times R^m$, where D is a domain in R^n.

Using Feynman-Kac formula, we can obtain from (3.2) that V satisfies the following PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}\frac{\partial^2 V}{\partial X_i X_j} + \mu_i \sum_i \frac{\partial V}{\partial X_i} - (\lambda + r)V = r - k - (R_2 V^+ - V^-)\lambda \quad (X,t) \in D \times [0,T) \tag{3.4}$$

$$V|_{t=T} = 0, \quad X \in D \tag{3.5}$$

Let $\lambda = 0$, we have the PDE which the risk swap U satisfies:

$$\frac{\partial U}{\partial t}+\frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}\frac{\partial^2 U}{\partial X_i X_j}+\mu_i\sum_i\frac{\partial U}{\partial X_i}-rU=r-k \quad (X,t)\in D\times[0,T) \tag{3.6}$$

$$U|_{t=T}=0, \quad X\in D \tag{3.7}$$

3.3 A model with CIR interest rates

As mentioned in the introduction, interest rates play an important role in valuation of the CVA or CCR in a standard interest swap. If interest rates are supposed to be constant, the swap is meaningless. We assume that the interest rates follow CIR model

$$dr_t=\kappa(\theta-r)dt+\sigma\sqrt{r_t}dW_t \tag{3.8}$$

With the Feller condition

$$2\kappa\theta > \sigma^2 \tag{3.9}$$

where κ , θ , σ are positive constant parameters. The parameter κ corresponds to the speed of adjustment, θ to the mean and σ to volatility. Under (3.9), r_t is a positive process.

Since the counterparty with CCR is the entity receiving a floating rate, we suppose that the hazard rate of her is negatively correlated to the interest rate. Thus, for the sake of simplicity, we assume that the hazard rate λ is of the form $\lambda = \frac{a}{r+c}+b$, where $a,b,c \geq 0$. Then, the PDE (3.4), (3.5) and (3.6), (3.7) on $(r,t)\in(0,\infty)\times[0,T)$ can be rewritten as

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 r\frac{\partial^2 V}{\partial r^2}+\kappa(\theta-r)\frac{\partial V}{\partial r}=r-k-(R_2V^+-V^-)(\frac{a}{r+c}+b)+(\frac{a}{r+c}+b+r)V \tag{3.10}$$

$$V|_{t=T}=0 \quad 0<r<\infty \tag{3.11}$$

$$\frac{\partial U}{\partial t}+\frac{1}{2}\sigma^2 r\frac{\partial^2 U}{\partial r^2}+\kappa(\theta-r)\frac{\partial U}{\partial r}=r-k+rU \tag{3.12}$$

$$U|_{t=T}=0 \quad 0<r<\infty \tag{3.13}$$

Remark 3.1 Equations (3.10) and (3.12) are degenerate on $r=0$. Since $2\kappa\theta > \sigma^2$, according to Fichera's theory [9], we can not impose any boundary conditions on $r=0$. Therefore, the above Cauchy terminal problems (3.10), (3.11) and (3.12), (3.13) are well-posed.

4. Iteration approaches

PDE (3.12) is a linear equation, therefore many numerical methods can be applied to it, no matter whether we can obtain the analytical solution. However, PDE (3.10) is a nonlinear equation. Due to the nonlinearity, it is difficult to obtain its solution even though by the numerical approach. Motivated by the contraction mapping principle, we will have an iteration approach which satisfies:

- We can obtain the solution (numerical or analytical) step by step by solving a linear PDE.
- The iteration has to converge to the solution of (3.10).

We denote the iteration result of step $i(\geq 0)$ by V_i .

When $i=0$, we obtain V_0 by solving (3.12) and (3.13), that is $V_0 = U$. When $i = n+1$, we obtain V_{n+1} by solving the following PDE

$$\frac{\partial V_{n+1}}{\partial t} + \frac{1}{2}\sigma^2 r \frac{\partial^2 V_{n+1}}{\partial r^2} + \kappa(\theta - r)\frac{\partial V_{n+1}}{\partial r} = r - k - (R_2 V_n^{\ +} - V_n^{\ -})(\frac{a}{r+c} + b) + (r + \frac{a}{r+c} + b)V_{n+1} \tag{4.1.n}$$

$$V_{n+1}\,|_{t=T} = 0 \quad 0 < r < \infty \tag{4.2.n}$$

where V_n has been obtained in last iteration step.

Remark 4.1 in the iteration steps, we can see that the value of the swap with CCR in risk free closeout is the result of the iterations' first step, i.e. V_1 is the swap's value in risk free closeout. Thus, considering the definition of CVA in section 2, the CVA computed in conventional closeout is the first order approximation of the CVA in risky closeout.

To prove the convergence of the iteration, we have

Theorem 4.1 Sequence $\{V_n\}_{n\geq 0}$ is monotonically decreasing.

Theorem 4.2 The iterations converge to the solution of (3.10), that is

$$V = \sum_{n=0}^{\infty}(V_{n+1} - V_n) + V_0$$

The proofs of theorem 4.1 and 4.2 are shown in appendix B.

Remark 4.2 Since the CVA in risk free closeout is the first order approximation of the CVA in risky closeout, considering theorem 4.1, we have that the computation in conventional closeout underestimates the counterparty credit risk.

5. Numerical results

In this section, we will present numerical examples for pricing CVA by the iterations provided in section 4. The PDE in each iteration is solved numerically by Crank—Nicolson finite difference scheme. By the results, we try to achieve the following aims:

- Showing the impact of interest rates on CVA;
- Revealing the convergence of the iteration approach;
- Presenting the difference between CVAs computed in risky and risk free closeouts.

The parameters in the all figures are shown as follows:

$T = 3, R_1 = 0.4, a = 0.05, b = 0.05, c = 0.001, k = 0.05, \kappa = 0.15, \theta = 0.08, \sigma = 0.1$.

<Figure 1 about here>

In figure 1, we try to analyze the relation between the CVA and interest rates.

The figure shows that the CVA of a standard interest swap with CCR is negatively correlated to interest rates. When interest falls down, the counterparty receiving a floating rate is inclined to default, thus the investor is subject to relatively serious CCR.

<Figure 2 about here>

In figure 2, the convergence of the iterations provided in section 4 is shown. We can see that when the number of the iterations is large enough (in this example the number is 5), the results are almost invariant. The speed of convergence is fast.

<Figure 3 about here>

In figure 3, we can see that the CCR computed in conventional closeout (risk free closeout) is underestimated, compared to the CCR in risky closeout. When the CCR is in a high level, the difference is more obvious.

6. Conclusions

In this paper, we provide risky closeout amount in estimating counterparty credit risk. With this new type of closeout, we establish a nonlinear PDE model for pricing CVA in a standard interest swap with CCR under reduced form framework.

To solve the nonlinear PDE, we provide an iteration approach and prove the validity of the method. In the iteration procedure, we can see that the CVA computed in risk free closeout, which is usually applied in past papers, is the first order approximation of the CVA in risky closeout, and thus underestimates the CVA.

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Notes

Note1. Besides our work, Brigo and Morrini (2010) have focused on closeout amount. They provide a ‘substitution closeout’ to replace conventional closeout amount.

Note2. A risk free closeout amount is a net present value that assumes that the defaulted transaction is without CCR.

Note3. It can be seen that the Feller condition (3.9) is essential to our proof.

Note4. $\|\bullet\| = \sup|\bullet|$

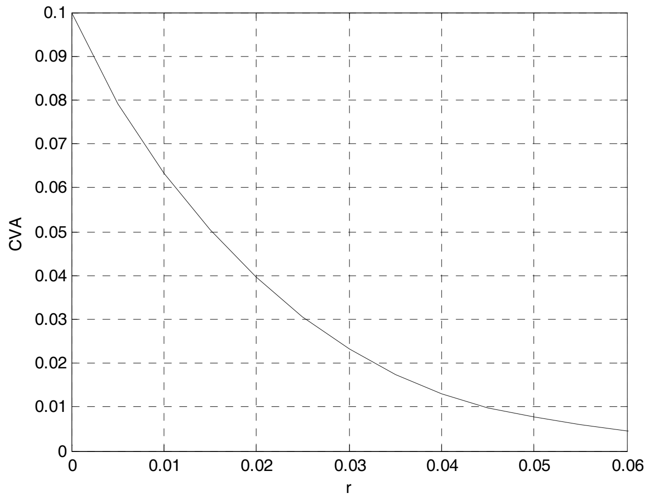

Figure 1 CVA with interest rates

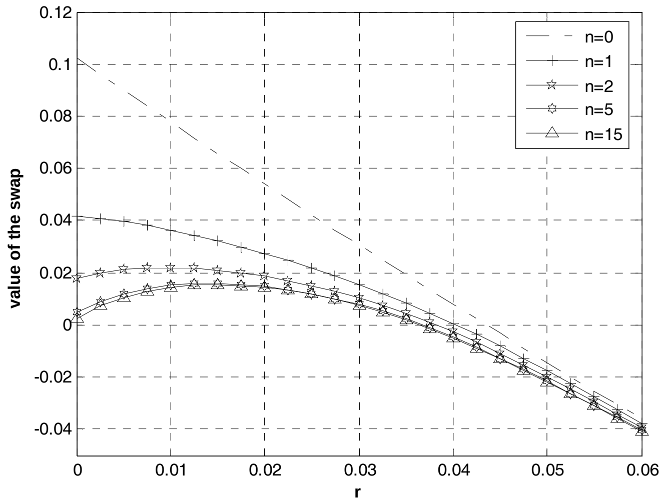

Figure 2 Convergence of the iterations

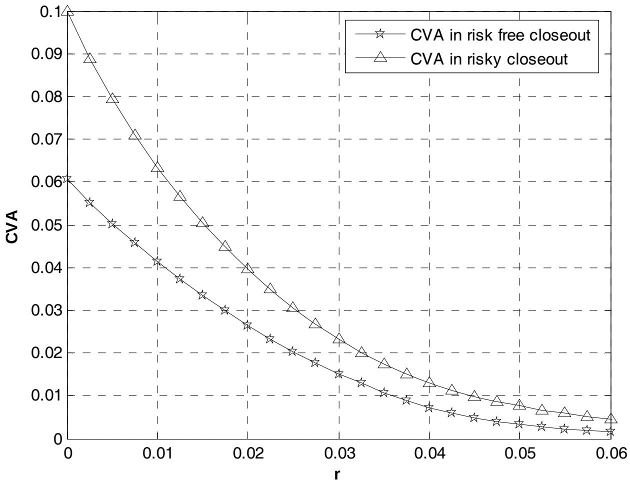

Figure 3 Difference between risky closeout and risk free closeout

Appendix

Appendix A proof of theorem 3.1

We separate the above formula into three parts, let

$$I_1 = E[k\int_t^T I_{\tau>s} e^{-\int_t^s r_\theta d\theta} ds \mid G_t],$$

$$\begin{aligned} I_2 &= E[I_{\tau>t}(e^{-\int_t^{\tau\wedge T} r_\theta d\theta} - 1) \mid G_t] \\ &= E[I_{\tau>t}(I_{t<\tau<T} e^{-\int_t^{\tau} r_\theta d\theta} + I_{\tau\geq T} e^{-\int_t^{T} r_\theta d\theta} - 1) \mid G_t] \end{aligned}$$

$$I_3 = E[(RV^+(\tau) - V^-(\tau)) e^{-\int_t^{\tau} r_\theta} I_{t<\tau<T} \mid G_t].$$

According to the Lemma 5.1.2 in the reference [2],

We have

$$I_1 = I_{\tau>t} \frac{E[k\int_t^T I_{\tau>s\geq t} e^{-\int_t^s r_\theta d\theta} ds \mid F_t]}{E[I_{\tau>t} \mid F_t]},$$

$$I_3 = I_{\tau>t} \frac{E[(RV^+(\tau) - V^-(\tau)) e^{-\int_t^{\tau} r_\theta d\theta} I_{t<\tau<T} \mid F_t]}{E[I_{\tau>t} \mid F_t]}.$$

According to the standard assumptions in reduced form framework, we have

$$I_1 = I_{\tau>t} E[k\int_t^T e^{-\int_t^s (\lambda_\theta + r_\theta) d\theta} ds \mid F_t].$$

Divide $[t,T]$ into N intervals:

$$t = s_0 < s_1 < \ldots < s_N = T\ ,$$

where, without loss of generality, we take $s_j = t + j\Delta s$ and $\Delta s = \frac{T-t}{N}$. Then

$$\begin{aligned}
I_3 &= I_{\tau>t}\frac{\sum\limits_{t<s_i<T} E[(RV^+(s_i) - V^-(s_i))e^{-\int_t^{s_i} r_\theta d\theta} I_{s_i<\tau<s_{i+1}} \mid F_t]}{E[I_{\tau>t} \mid F_t]},\\
&= I_{\tau>t}\frac{\sum\limits_{t<s_i<T} E[(RV^+(s_i) - V^-(s_i))e^{-\int_t^{s_i} r_\theta d\theta}(e^{-\int_t^{s_i}\lambda_\theta d\theta} - e^{-\int_t^{s_{i+1}}\lambda_\theta d\theta}) \mid F_t]}{e^{-\int_0^t \lambda_\theta d\theta}},\\
&= I_{\tau>t}E[\int_t^T (RV^+(s) - V^-(s))\lambda e^{-\int_t^s(\lambda_\theta + r_\theta)d\theta} ds \mid F_t].
\end{aligned}$$

Using the same method to I_2, we have

$$I_2 = I_{\tau>t}E[\int_t^T \lambda_s e^{-\int_t^s (r_\theta+\lambda_\theta)d\theta} ds + e^{-\int_t^T(\lambda_\theta+r_\theta)d\theta} - 1 \mid F_t].$$

Since

$$V = I_1 + I_2 + I_3\ ,$$

Then

$$V_t = I_{\tau>t}E[\int_t^T ke^{-\int_t^s(\lambda_\theta+r_\theta)d\theta}ds + \int_t^T \lambda_s e^{-\int_t^s(r_\theta+\lambda_\theta)d\theta}ds + e^{-\int_t^T(\lambda_\theta+r_\theta)d\theta} - 1 + \int_t^T \lambda_s e^{-\int_t^s(r_\theta+\lambda_\theta)d\theta}(RV^+ - V^-)ds \mid F_t]$$

This completes the proof.

Appendix B Proofs of theorem 4.1 and 4.2

Definition B.1

$$f(x) = R_2 x^+ - x^- \quad x \in R$$

It can be seen that $f(x)$ is monotonically increasing and Lipchitz continuous.

We suppose $Q = (0,\infty)\times[0,T]$.

We let L and L_1 be a differential operator defined by

$$Ly \equiv \frac{\partial y}{\partial t} + \frac{1}{2}\sigma^2 r \frac{\partial^2 y}{\partial r^2} + \kappa(\theta - r)\frac{\partial y}{\partial r} - h(r)y \cdot$$

$$L_1 y = Ly - y \cdot$$

Where $h(r) \geq 0$.

Lemma B.1 For $v \in C^{2,1}(Q_1) \cap L_\infty(Q)$, if v satisfies

$$Lv \geq 0 \tag{B.1}$$

$$v\,|_{t=T} \leq 0 \tag{B.2}$$

With the Feller condition $2\kappa\theta > \sigma^2$.

Then we have

$$v \leq 0 \text{ for any } (r,t) \in Q$$

Proof

Introduce the transformation $u = e^t v$, and then we have $L_1 u \geq 0$ on Q and $u \leq 0$ on $t = T$.

We denote $\sup_Q |u|$ by A. For $\delta > 0$, $B > 0$, let $Q_{\delta B} = \{(r,t) \mid \delta \leq r \leq B, 0 \leq t \leq T\}$.

Consider the auxiliary function $w = u - A(\frac{r+C}{B} + \frac{\delta^\varepsilon}{r^\varepsilon})$ where $C > 0$ $\varepsilon > 0$.

By simple calculation, we have

$$L_1 w = L_1 u - A(\frac{1}{B}(\kappa(\theta - r) - (h(r)+1)(r+C)) + \delta^\varepsilon(\frac{1}{2}\sigma^2 r\varepsilon(\varepsilon+1)\frac{1}{r^{\varepsilon+2}} - \kappa(\theta - r)\varepsilon\frac{1}{r^{\varepsilon+1}} - (h(r)+1)\frac{1}{r^\varepsilon}))$$

$$\geq A[\frac{1}{B}(-\kappa\theta + C) + \frac{\delta^\varepsilon}{r^{\varepsilon+1}}((1-\kappa\varepsilon)r + \varepsilon(\kappa\theta - \frac{1}{2}\sigma^2 - \frac{1}{2}\sigma^2\varepsilon))]$$

in $Q_{\delta B}$

(B.3)

To guarantee $L_1 w > 0$ in $Q_{\delta B}$, we let $C > \kappa\theta$, $0 < \varepsilon < \min\{\frac{\kappa\theta - \frac{1}{2}\sigma^2}{\frac{1}{2}\sigma^2}, \frac{1}{\kappa}\}$ (Note 3).

And w is non positive on the whole parabolic boundary

$\{r = B, 0 \leq t \leq T\}$, $\{r = \delta, 0 \leq t \leq T\}$, $\{\delta \leq r \leq B. t = T\}$.

By standard argument, we claim

$$w \leq 0 \quad \text{on } Q_{\delta B}$$

That is

$$u \leq A(\frac{r+C}{B} + \frac{\delta^\varepsilon}{r^\varepsilon}) \quad \text{on } Q_{\delta B}$$

For any fixed $(r_0, t_0) \in Q_1$, when B is large enough and δ is small enough, we obtain $(r_0, t_0) \in Q_{\delta B}$.

Let $B \to \infty$ $\delta \to 0$, we have $u(r_0, t_0) \leq 0$ for any $(r_0, t_0) \in Q$.

Considering $e^{-t}u = v$, we have $v(r,t) \leq 0$ on Q.

This completes the proof.

Remark B.1 The lemma shows that without any conditions on the boundary $r = 0$, the maximum principle is still valid for equation (3.10)

Proof of theorem 4.1

We denote $V_n - V_{n-1}$ by I_n. ($n = 1,2,3...$).

It can be seen that I_n is a solution of the following problem

$$L_0 I_n = g_n(r,t) \quad (r,t) \in Q \tag{B.4}$$

$$I_n|_{t=T} = 0 \quad r \in (0,\infty) \tag{B.5}$$

Where $$L_0 I_n = \begin{cases} \dfrac{\partial I_n}{\partial t} + \dfrac{1}{2}\sigma^2 r \dfrac{\partial^2 I_n}{\partial r^2} + \kappa(\theta - r)\dfrac{\partial I_n}{\partial r} - rI_n, n = 1 \\ \dfrac{\partial I_n}{\partial t} + \dfrac{1}{2}\sigma^2 r \dfrac{\partial^2 I_n}{\partial r^2} + \kappa(\theta - r)\dfrac{\partial I_n}{\partial r} - (\lambda + r)I_n, n \geq 2 \end{cases} \tag{B.6}$$

$$g_n(r,t) = \begin{cases} (a_2 r + b_2)(1 - R_2)V_0^+ \quad n = 1 \\ (a_2 r + b_2)(f(V_{n-2}) - f(V_{n-1})) \quad n > 1 \end{cases} \tag{B.7}$$

when $n = 1$, due to $g_1(r,t) \geq 0$, according to lemma B.1, we have

$$I_1 = V_1 - V_0 \leq 0 \tag{B.8}$$

We prove the statement by induction.

If theorem 4.1 is valid for $n = k - 1$, which means

$$I_{k-1} = V_{k-1} - V_{k-2} \leq 0 \tag{B.9}$$

then since f is monotonically increasing, we have

$$g_k(r,t) \geq 0 \tag{B.10}$$

By using lemma B.1, we have

$$I_n \leq 0 \tag{B.11}$$

This completes the proof.

Proof of theorem 4.2

Introduce the transformation

$$e^{-t}H = V \tag{B.12}$$

$$e^{-t}H_n = V_n \tag{B.13}$$

Thus $e^{-t}J_n = I_n$, where $J_n = H_n - H_{n-1}$, $n = 1,2,3...$

since $I_n \le 0$, we have $J_n \le 0$.

Considering (4.1.n) and (4.2.n), we have

$$\frac{\partial J_{n+1}}{\partial t}+\frac{1}{2}\sigma^2 r\frac{\partial^2 J_{n+1}}{\partial r^2}+\kappa(\theta-r)\frac{\partial J_{n+1}}{\partial r}-(\lambda+r+1)J_{n+1}=\lambda(f(H_{n-1})-f(H_n)) \tag{B.14}$$

We denote $\sup(\frac{\lambda}{\lambda+r+1})\inf_Q J_n$ by E.

We have

$$\begin{aligned}&\frac{\partial(J_{n+1}-E)}{\partial t}+\frac{1}{2}\sigma^2 r\frac{\partial^2(J_{n+1}-E)}{\partial r^2}+\kappa(\theta-r)\frac{\partial(J_{n+1}-E)}{\partial r}-(\lambda+r+1)(J_{n+1}-E)\\&=\lambda(f(H_{n-1})-f(H_n))+(\lambda+r+1)E\\&\le-\lambda J_n+(\lambda+r+1)\sup(\frac{\lambda}{\lambda+r+1})\inf_Q J_n\\&\le\lambda(-J_n+\inf_Q J_n)\le 0\end{aligned}$$

Since $J_{n+1}-E\,|_{t=T}\ge 0$,

according to lemma B.1,

We have $J_{n+1}-E\ge 0$ for $(r,t)\in Q$

Thus $\inf_Q J_{n+1}\ge\sup_Q\left(\frac{\frac{a}{r+c}+b}{\frac{a}{r+c}+b+r+1}\right)\inf_{Q_1} J_n=\left(\frac{\frac{a}{c}+b}{\frac{a}{c}+b+1}\right)\inf_{Q_1} J_n$.

Since $J_n\le 0$,

we have $\|J_{n+1}\|\le\left(\frac{\frac{a}{c}+b}{\frac{a}{c}+b+1}\right)\|J_n\|$ (Note 4).

By the contraction mapping principle, H_n converges to H when $n\to\infty$, that is

$$H=\sum_{n=0}^{\infty}(H_{n+1}-H_n)+H_0$$

Considering (B.12) and (B.13), we have

$$V=\sum_{n=0}^{\infty}(V_{n+1}-V_n)+V_0.$$

This completes the proof.