Credit Risk Measurement Based on the Markov Chain

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Abstract

Credit migration matrices are often used in many credit risk and pricing application, and typically assumed to be generated by a simple Markov process. This paper is going to analyze the basic elements of credit risk research, and Maximum Likelihood estimation will be adopted to estimate the Mover-Stayer model's parameters in this paper. Furthermore, the recursive method will be used to compute the Maximum Likelihood estimator, and the numerical results can illustrate the strength of the Mover-Stayer model on credit risk analysis. We also use the hypotheses to prove that the Markov chain suit for the data against the hypotheses that the Mover-Stayer model more suitable for the data. Finally, we will make some comparisons according to the output of the program, and obtain some conclusions. The Mover-Stayer Model is more suitable against according the numbered result.

Keywords: Credit Risk, Markov Chain, Mover-Stayer model, Maximum Likelihood, Probabilities

1. Introduction

Credit risk is an ancient global matter. Stiglitz and Weiss pointed out that because of information asymmetry is one of major reason for credit risk in the financial market, high-risk businesses are willing to apply for a loan with higher costs, but the banks can not effectively identify the level of enterprise risk. Therefore, the interest rate can not act as leverage between supply and demand. The supply of credit markets will be less than the demand, making some enterprises unable to receive a loan. As a result, "credit rationing" comes into being in bank loans, which is also an important reason why credit risk bores. Due to the complexity and uncertainty of credit risk research, measurement and management of credit risk became more and more important and difficult. For the large number of bank practices, government supervisors and scholars, credit risk research on one hand is challenging and on the other hand is highly rewarded. The beginning of my dissertation is going to analyze the basic elements of credit risk research, clearing the way for further research of its concept.

The measurement of credit risk has changed dramatically in the past few years, Many Scholars focus on this field. Linda Allen and Anthony Saunders (2004) find that the possibility of systematic correlation between the probability of default and loss given default is also neglected in currently available models. Barnhill and Gleason (2001) consider the correlation of credit and interest rate risk. They show that correlations across risk exposures can cause an increase in failure probability for a bank with a positive duration gap. That is, increases in credit risk exposure are accompanied by increases in interest rate risk exposure. Dean Fantazzini, Silvia Figini (2009) also find that non-parametric model performs much better that the classical logit model. As for the out-of-sample performances, the evidence is just the opposite, and the logit performs better than the RSF model. They explained this evidence by showing how error in the estimates of default probabilities can affect classification error when the estimates are used in a classification rule.

Maximum Likelihood estimation will be first adopted to estimate the Mover-Stayer model's parameters in our paper. Furthermore, the recursive method can then be used to easily compute the Maximum Likelihood estimator, and the numerical results can illustrate the strength of the Mover-Stayer model on credit risk analysis. We can also use the hypotheses to prove that the Markov chain suit for the data against the hypotheses that the Mover-Stayer model more suitable for the data. Finally, we will make some comparisons according to the output of the program, and obtain some conclusions. The Mover-Stayer Model is more suitable against according the numbered result.

2. The Mover-Stayer Model

2.1 The introduction of the Mover-Stayer Model

The Mover-Stayer model was firstly founded by I.Blumen, M.Kogan and P.J.McCarthy (referred to as BKM herein) in their interesting study of the moment of workers among various industrial aggregates in the U.S. The Mover-Stayer model could be described as a generalization of the Markov chain model, and the mover-stayer model actually is a discrete time under consideration, stochastic process {Z (*j*), *j*>=0} which includes the two independent Markov chain. We use an example to illustrate the mover-stayer model. A button factory is grouped into a finite number, *I*, of factory code categories. In the *ith* code category (*i=1, 2, 3, 4, 5……, I*), there are about two kinds of workers which is the stayer and movers. Let S_i denotes the proportion of workers in the

*i*th code category in the initial quarter who are the stayers (*i=1, 2, 3, 4, 5……,* I). So 1-S_i is the proportion of workers in the *i*th code category in the initial quarters who are the movers. We can assume that each stayer is always in a special code category with probability one, and that each mover can always change its code category all the time, and it can be described by a one-quarter transition probability matrix which is listed in the below:

$$\mathbf{M} = \left\{ \begin{array}{cccc} m_{11} & m_{12} & \cdots & m_{1I} \\ m_{21} & m_{22} & \cdots & m_{2I} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ m_{I1} & m_{I2} & \cdots & m_{II} \end{array} \right\}$$

where m_{ij} is the probability that the mover who is in the *i*th code category in a special quarter and will be in the *j*th category (*i, j =1, 2, 3 …, I*) in the following quarter. However, the behavior of the stayers can be described by a transition probability matrix which is located in the diagonal cells (where *i=j*). The remaining cells represent the movers which can be described by the Markov chain.

The transition probability matrix *M* which describes the activity of the movers, and S_i stands for the proportion of the workers in each code category in the initial quarter that stands for the stayers (*i=1,2,3,………,I*), of course, which can not be recognized in the matrix, that is to say, which are unknown parameters in the matrix. Then we let S={ $S_{i,}$ $i \in s$ }, at the same time the 0<=S_i<1 and the m_{ij}>0, for the *i, j=1, 2, 3,……, I*. However, the parameter can not be estimated directly according to the data which is used by BKM, because there is no way of determining unequivocally which workers in a given code categories in a specified quarter stayers and which are movers.

Additionally, we define the numbers of the length of the times points *J*. Actually we have *J+1* numbers of the length of the times points. We define the *J* in order to simplify the Fellow method that is the recursive method. The formula is defined as followed:

J+1= the number of the length of the time points.

The above equation, that is to say that the data can be considered as panel data collected at *J+1* time points.

2.2 Maximum Likelihood Estimation

According to the above descriptions of the mover-stayer model, the maximum likelihood estimation method is used to solve the log-likelihood equations. As shown, we will use the maximum likelihood to solve many equations as stated. However, there are some unknown parameters in the equation which are listed in the follow. The remaining estimators are derived from the estimators of m_{ii}'s in a recursive manner. Before we write the likelihood function, we need to define some notations which are listed as followed:

Where,

1) The initial probabilities η_j : the number of such probabilities: η_1 η_2.....,η_j .

2) The probability or portion of the stayers S_i The Probabilities or proportion of the movers $(1-S_i)$.

3) The number of such probabilities: $n_i(0)$-n_i stands for the initial states under the situation of the mover apart from

the situation of the stayers.

4) Probabilities for paths with states only i $[S_i+(1-S_i)m_{ii}^J]$ stand for the total number of such probabilities or proportions n_i .

5) The probabilities or portions of the transitions from i to i m_{ii} stands for the number of the probabilities under the mover situations.

6) Probabilities of transitions from i to k ($i \neq k$): m_{ik} stands for the number of such probability or proportion n_{ik} .

7) $n_i^* = \sum_{j=1}^{J} n_i(j-1)$ stands for the total number of visiting to state i to state i before time J

According to the above descriptions, where

$$\sum_{i=1}^{w} n_i(j) = \sum_{(io \dots ij) \in s^{J+1}} n_{io,\dots\dots ij} = n$$

Of course, the likelihood function is obtained by noting that the assumption of individual histories are independent of each other it implies that random variables $\{=(1-s_{io})\eta_{io}m_{ioio}{}^{J}\dots m_{ij-1ij},\dots ij,\ (i_0,\dots i_j)\in S^{J+1}\}$ given by the joint distributions of the mover-stayer model. That is

$$\begin{aligned}\pi_{io,\dots,ij} &= P[Z(0)=i_0, Z(1)=i_1,\dots,Z(J)=i_j] \\ &= s_{io}\eta_{io} + (1-s_{io})\eta_{io}m_{ioio}{}^{J} \qquad \text{If } i_0=i_1=\dots=i_j \\ &= (1-s_{io})\eta_{io}m_{ioio}{}^{J}\dots m_{ij-1ij} \qquad \text{Otherwise,}\end{aligned}$$

where $\{\eta_i \geq 0, 1\leq i\leq w, \sum_{i=1}^{w}\eta_i=1\}$ denotes an initial distribution of $\{Z(j), j\geq 0\}$, $i_j\in s, 0\leq j\leq J$. Thus the likelihood function $L\equiv(\eta, M, s)$ up to the multiplicative constant of the form

$$L = [\pi_{io\dots\dots ij}]^{n_{io\dots\dots ij}}$$

$$= \prod_{i=1}^{w}[s_i\eta_i + (1-s_i)\eta_i m_{ii}^{J}]^{ni} \prod_{(i_o,i_1\dots\dots i_J)\in s^{J+1}} \times [(1-S_{IO})\eta_{IO}m_{IOI1},\dots m_{ij-1ij}]^{n_{io\dots\dots ij}}$$

For each state i, we will group together the factors in order to involve the state $i,$ so we can obtain the formula as followed:

$$\begin{aligned}L &= \prod_{i=1}^{w}\eta_i^{n_i(0)}\prod_{i=1}^{w}[S_i+(1-S_i)m_{ii}{}^{J}]^{ni}\times\prod_{i=1}^{w}(1-S_i)^{n_i(0)-n_i}\prod_{i=1}^{w}m_{ii}{}^{n_{ii}-Jn_i}\prod_{i\neq k}^{w}m_{ik}^{n_{ik}} \\ &= \prod_{i=1}^{w}\eta_i^{n_i(0)}\prod_{i=1}^{w}L_i \qquad (1)\end{aligned}$$

where

$$L_i = [s_i + (1-s_i)m_{ii}^{J}]^{n_i}(1-s_i)^{n_i(0)-n_i}\, m_{ii}^{n_{ii}-Jn_i} \times \prod_{k=1,k\neq i}^{w} m_{ik}{}^{n_{ik}}, i\in S,$$

Because according to the above equation, we can obtain the formula which is listed as followed:

$$L = \prod_{i=1}^{w}\eta_i^{n_i(0)}\prod_{i=1}^{w}L_i$$

$$\log L = \sum_{i=1}^{w-1} n_{i(0)}\log\eta_i + n_w(0)\log(1-\sum_{i=1}^{w-1}\eta_i) + C$$

Now we define the $\prod_{i=1}^{w}\log L_i = C$

We change the format of the above equation as followed:

$$\log L = \sum_{i=1}^{w-1} n_{i(0)} \log \eta_i + n_w(0) \log(1 - \sum_{i=1}^{w-1} \eta_i) + C$$

$$\frac{\partial \log L}{\partial \eta_1} = 0$$

We can obtain the below formula:

$$\frac{n_1(0)}{\eta_1} = \frac{n_w(0)}{1 - \sum_{i=1}^{w-1} \eta_i}$$

At the same time we can get the $\eta_1 \ \eta_2 .....,\eta_j$

We can get the recursion is listed as followed:

$$\eta_j = \frac{\eta_j(0)}{\sum_{i=1}^{w} n_i(0)}$$

At the same time, according to the formula (1), we can get the below parameters

$$\log L_i = n_i \log[s_i + (1-s_i) m_{ii}{}^J] + [n_i(0) - n_i] \log(1-s_i) + (n_{ii} - Jn_i) \log m_{ii} + \sum_{k=1,k\neq i}^{w} n_{ik} \log m_{ik} \quad (2)$$

In order to find critical points of the parameter L_i, firstly we set

$$\frac{\partial \log L_i}{\partial s_i} = \frac{n_i(1 - m_{ii}{}^J)}{s_i + (1-s_i) m_{ii}{}^J} - \frac{[n_i(0) - n_i]}{(1-s_i)} = 0$$

Solving the above equation, we can get s_i which is listed as followed:

$$s_i = [n_i - n_i(0) m_{ii}{}^J] / [n_i(0)(1 - m_{ii}{}^J)]$$

When we obtain the formula of the s_i in the above equation, and at the same time we denote it by C a quantity that does not depend on parameters, we can obtain the formula again

$$\log L_i = C - [n_i(0) - n_i] \log(1 - m_{ii}{}^J) + (n_{ii} - Jn_i) \log m_{ii} + \sum_{k=1,k\neq i}^{w} n_{ik} \log m_{ik}$$

In order to simplify the above formula, let we define

$$g_i(m_{ii}) \equiv [n_i(0) - n_i] \log(1 - m_{ii}{}^J) + (n_{ii} - Jn_i) \log m_{ii}$$

For $i \neq w$, we rewrite the formula $\log L_i$ as

$$\log L_i = C - g_i(m_{ii}) + \sum_{k=1,k\neq i}^{w-1} n_{ik} \log m_{ik} + n_{iw} \log(1 - \sum_{k=1}^{w-1} m_{ik}) \quad (3)$$

Of course, where $\sum_{k=1,k\neq i}^{w} n_{ik} \log m_{ik} = \sum_{k=1,k\neq i}^{w-1} n_{ik} \log m_{ik} + n_{iw} \log(1 - \sum_{k=1}^{w-1} m_{ik})$

And according to the above formula, we will first obtain the parameter $m_{i,w-1}$, we define at the beginning

$$\frac{\partial \log L_i}{\partial m_{i,w-1}} = \frac{n_{i,w-1}}{m_{i,w-1}} - \frac{n_{iw}}{1 - \sum_{k=1}^{w-1} m_{ik}} = 0$$

And we solve the above formula, we can get the $m_{i,w-1}$, whose formula is listed as follow:

$$m_{i,w-1} = \frac{n_{i.w-1} \times (1 - \sum_{k=1}^{w-1} m_{ik})}{n_{iw}}$$

$$m_{i,w-1} = \frac{n_{i.w-1} \times (1 - \sum_{k=1}^{w-1} m_{ik})}{n_{iw}} = \frac{n_{i,w-1}(1 - \sum_{k=1}^{w-2} m_{ik})}{\sum_{k=w-1}^{w} n_{ik}}$$

According to the above expression of the formula of the $m_{i,w-1}$, we can get at the same formula $\log L_i$ which is listed as followed:

$$\log L_i = C - gi(m_{ii}) + \sum_{k=1,k\neq i}^{w-2} n_{ik} \log m_{ik} + \sum_{k=w-1}^{w} n_{ik} \log(1 - \sum_{k=1}^{w-2} m_{ik})$$

According to the above formula, we can get the parameter $m_{i,w-2}$ which is listed as followed:

$$m_{i,w-2} = \frac{n_{i,w-2}(1 - \sum_{k=1}^{w-2} m_{ik})}{n_{i.w-1}}$$

We can use the same method to get the parameters $m_{i,w-2}, m_{i,w-3},\ldots\ldots\ldots\ldots m_{i,i+1}, m_{i,i-1},\ldots\ldots\ldots\ldots, m_{i1}$ in differentiating $\log L_i$ in turn with respect to this method, in that case, we can get a parameter by solving the above total equation every time. Finally, we use the go-back method to get all the parameters by taking the log-likelihood functions.

In the same procedure, we can obtain the formula as followed:

$$\frac{\partial \log L_i}{\partial m_{ij}} = \frac{n_{ij}}{m_{ij}} - \sum_{k=j+1,k\neq i}^{w} n_{ik} / (1 - \sum_{k=1}^{j} m_{ik}) = 0,$$

$j = w-1, w-2, \ldots\ldots, i+1, i-1, \ldots\ldots 1$, and solving for m_{ij}, finally we can get the formula as followed:

$$m_{ij} = n_{ij}(1 - m_{ii} - \sum_{k=1,k\neq i}^{j-1} m_{ik}) / \sum_{k=j,k\neq i}^{w} n_{ik}$$

When we get the above formula, we can change the format of the formula (3) which is listed as followed:

$$\log L_i = C - g_i(m_{ii}) + \sum_{k=1,k\neq i}^{j-1} n_{ik} \log m_{ik} + (\sum_{k=j}^{w} n_{ik}) \log(1 - m_{ii} - \sum_{k=1,k\neq i}^{j-1} m_{ik})$$

According to get the formula, we still have a situation which is not considered, that is when *j*=1 or *j*=2 if *i*=1). We obtain $\log L_i$ as the following function of m_{ii} only:

$$\log L_i = C - g_i(m_{ii}) + \sum_{k=1,k\neq i}^{w} n_{ik} \log(1 - m_{ii}) \tag{4}$$

i=1, 2, 3,…….., *w*-1. The above equations also hold for *i*=w, because if *i*=w we can rewrite (1) as follow:

$$\log L_w = C - g_w(m_{ww}) + \sum_{k=2}^{w} n_{wk} \log m_{wk} + n_{w1} \log(1 - \sum_{k=2}^{w} m_{wk})$$

We will apply the same procedure as for $i \neq w$ that is to say, differentiating $\log L_w$ with respect to m_{wj} in the order *j*=*w*-1,*w*-2,…….,1 we obtain $\log L_w$ as a function on m_{ww} only, as given by (3).

Finally, we set the formula as followed:

$$\partial \log L_i / \partial m_{ii} = 0, i \in s$$

And as we know that,

$$\sum_{k=1,k\neq i}^{w} n_{ik} = n_i^* - n_{ii}$$

We obtain the equation for m_{ii}, we take the log for the formula (4) $i \in s$,

$$[n_i^* - Jn_i(0)]m_{ii}^{J+1} + [Jn_i(0) - n_{ii}]m_{ii}^{J} + [Jn_i - n_i^*]m_{ii} + n_{ii} - Jn_i = 0 \quad (5)$$

We set the left side of the above equation by $f(m_{ii})$, and we can observe that

$$f(0) = n_{ii} - Jn_i = \sum_{j=1}^{J} [n_{ii}(j) - n_i] > 0, f(1) = 0$$

where

$$f^1(1-) = J[n_i^* - n_i(0) - n_{ii} + n_i] > 0$$

Furthermore, it can be seen from a simple computation and by drawing the picture of the convex function, that we find that the function $f(m_{ii})$ changes its convexity at most once in the interval (0,1). At the same time, we combine this fact with $f(0)$>0, $f(1)$=0, and $f^1(1-) > 0$ shows that the formula (5) has a unique root between the interval (0, 1). But if m_{ii} is the root of the function, then the corresponding m_{ij} 's, $j \neq i$, can be also computed recursively as

$$\hat{m}_{ij} = n_{ij}(1 - \hat{m}_{ii} - \sum_{k=1,k\neq i}^{j-1} \hat{m}_{ik}) / \sum_{k=j,k\neq i}^{w} n_{ik}, \quad i,j \in S \quad (6)$$

We begin with the lowest value of *j*, that is, if *j*=1 but $i \neq 1$, and if *j*=2 and *i*=1. Then the corresponding value of the $\hat{s}_i$ is given by

$$\hat{s}_i = (n_i - n_i(0)\hat{m}_{ii}^{J}) / n_i(0)(1 - \hat{m}_{ii}^{J}), i \in S \quad (7)$$

In order to give a clearly interpretation of the estimation $\hat{s}_i$, we can rewrite it as followed:

$$\hat{s}_i = 1 - (n_i(0) - n_i) / n_i(0)(1 - \hat{m}_{ii}^{J})$$

The formula $n_i(0) - n_i$ stands for the number of individuals that start in state *i* and makes at least one transition between the *J* periods. The formula $n_i(0)(1-\hat{m}_{ii}^{J})$ is the expected number of such individuals, at the beginning we assume that all the $n_i(0)$ individuals stand for the movers according to transition matrix $\hat{M}$. So the difference between 1 and the ratio of the observed stands for the expected number of the individuals $\hat{s}_i$ which represent the estimate of the proportion of stayers according to transition matrix $\hat{M}$.

Next, we will derive Goodman's estimators by using the formula (5). For obtaining the first estimator, we set the $J \to \infty$, so the formula (5) can be changed as

$$[Jn_i - n_i^*]m_{ii} = Jn_i - n_{ii}, i \in S \quad (8)$$

The solution of the above equation is listed as followed:

$$\hat{m}_{ii} = (n_{ii} - Jn_i) / (n_i^* - Jn_i), i \in S \quad (9)$$

And at the same time, $\overline{m}_{ij}$ is computed according to the formula (6). By setting $J \to \infty$ in the formula (7), the estimator of s_i is obtained as followed:

$$\overline{s}_i = n_i / n_i(0) \quad (10)$$

$\overline{M}$ and $\overline{s}$ were expressed by the Goodman without derivation as maximum likelihood estimators of *M* and *s* respectively. It is obviously from the preceding of the derivation that they will be in accordance with the maximum likelihood estimators $\overline{M}$ and $\overline{s}$ just only when *J*= {the number of time periods} is very large. The estimator $\overline{s}_i$ is the simple numbers of the individuals who stands for the stayers. In this case, the estimator is more appealing when *J* is very large, because it is reasonable that we assume all individuals are the stayers in their states.

In order to obtain the second parameter of Goodman's estimators, first divide (4) by $Jn_i(0)$:

$$[\frac{n_i^*}{Jn_i(0)}-1]m_{ii}^{J+1}+[1-\frac{n_{ii}}{Jn_i(0)}]m_{ii}^{J}+[\frac{n_i}{n_i(0)}-\frac{n_i^*}{Jn_i(0)}]m_{ii}+\frac{n_{ii}}{Jn_i(0)}-\frac{n_i}{n_i(0)}=0 \quad (11)$$

At the same time we set that

$$n_i^* / Jn_i(0) = 1 \quad (12)$$

Then the formula (11) will be changed as followed:

$$[1-\frac{n_{ii}}{n_i^*}]m_{ii}^{J}+[\frac{n_i}{n_i(0)}-1]m_{ii}+\frac{n_{ii}}{n_i^*}-\frac{n_i}{n_i(0)}=0 \quad (13)$$

We set the parameter $\overline{m}_{ii}$ to denote the solution of the equation, and we can obtain the parameters $\overline{m}_{ij}$'s and $\overline{s}$ by computing from the formula (6) and the formula (7). Goodman derived (13) with a two-steps, and showed that *M* and *s* are the same with the estimators of *M* and $\overline{s}$ when the formula comes into existence, which is listed as followed:

$$(1/J)E[n_i^* \mathrm{I} n_i(0)] = n_i(0) \quad (14)$$

Because the parameter $\overline{m}_{ii}$ is a consistent estimator of m_{ii}, the derivation of formula (13) from the formula (11) shows that a little weaker condition,

$$n_i^* / Jn_i(0) \to 1 \quad (15)$$

There is required to be consistent with $\hat{m}_{ii}$. Under any conditions, because there is no reason to suppose that the formula (15) come into existence, generally, the estimator $\hat{m}_{ii}$ will not consistent. As we know that the estimators $\hat{m}_{ii}$ and $\hat{m}_{ii}$ will be greatly close to each other when $n_i^* / Jn_i(0) \approx 1$.

2.3 Hypothesis testing

According to the above method, we can give an example that is the application of the maximum likelihood estimator. In modeling panel data from some empirical process, we want to know that whether the mover-stayer model presents a better description of data than the Markov chain model. Actually, we can use the likelihood ratio principle to the test that difference between the two models. Observation of the stationary Markov chain is nested in the mover-stayer model; the mover-stayer model actually is a mixture of the Markov chain model. That is to say, it is better by using the mover-stayer model, in this situation, we set *s*= 0. Therefore, the form of the testing is H_o: $s = 0$ versus $H_A: s \neq 0$, where the equality s = 0 and inequality $s \neq 0$ are to be understood in the vector sense, and which can be based on the likelihood ratio statistic

$$\Lambda = \sup_{M,s=0} L(M,s) / \sup_{M,s} L(M,s) = L(\hat{p},0)/L(\hat{M},\hat{s})$$

Where $\hat{p} = \left\| n_{ik} / n_i^* \right\|$ is the maximum likelihood estimator of *M* under the hypothesis H_o; that is to say, it is a maximum likelihood estimator of a transition probability matrix of a Markov chain, and $\hat{M}$ and $\hat{s}$ can obtain by the formula (5) (6) and (7). So

$$L(\hat{p},0) = \prod_{i=1}^{w} (\frac{n_i(0)}{n})^{n_i(0)} \prod_{i,k} (\frac{n_{ik}}{n_i^*})$$

And

$$L(\hat{M},\hat{s}) = \prod_{i=1}^{w} \{(\frac{n_i}{n})^{n_i(0)}(\frac{n_i(0)-n_i}{n_i(0)})^{n_i(0)-n_i}\hat{m}_{ii}^{n_{ii}-Jn_i}\}\prod_{i \neq k} \hat{m}_{ik}^{n_{ik}}$$

Under H_o the asymptotic distribution of $-2\log\Lambda$ is x^2 with *w* degrees of freedom.

We can know that Goodman also considered the above test. However, his testing statistics, which differs from my method, were not derived from the likelihood ratio principle. With the above hypothesis and estimation for parameter of the Mover-Stayer Model, we will naturally check whether the Move-Stayer model can describe the data accurately and judge the Mover-Stayer Model how many probabilities for prediction by *n* independent realizations of the model with known parameters which can be totally the same as the true sequence.

3. Numerical comparison of the estimations

According to the above method, and at the same time, we can make some comparisons. As we know that we have three variables, which are respective, the number of the sequences, the number of the transitions in each sequence, and the range of the rating. We can set any numbers of the above three variables to see the different changes, that is to say, we make some sensitive analysis to choose an optimal result, and finally we analyze the reason of the optimal result.

For example, we set the number of the sequence equals to the 100, the range of the rating to 8, and the number of the transitions in each sequence to 29. The result is listed as followed:

$$\text{stayer} = \begin{bmatrix} 0.0900 & 0.0800 & 0.0700 & 0.0600 & 0.0500 & 0.0400 & 0.0300 & 0.0200 \end{bmatrix}$$

$$mover = \begin{bmatrix} 0.2730 & 0.1546 & 0.1716 & 0.0261 & 0.0838 & 0.1578 & 0.0573 & 0.0712 \\ 0.0538 & 0.2199 & 0.2134 & 0.0472 & 0.2169 & 0.0046 & 0.0441 & 0.2002 \\ 0.1341 & 0.1360 & 0.2011 & 0.0439 & 0.1030 & 0.1505 & 0.0427 & 0.1886 \\ 0.0909 & 0.1480 & 0.1671 & 0.2063 & 0.0783 & 0.0709 & 0.1275 & 0.1110 \\ 0.1741 & 0.1800 & 0.0113 & 0.0532 & 0.2629 & 0.1624 & 0.0591 & 0.0970 \\ 0.1518 & 0.1470 & 0.0703 & 0.0396 & 0.1046 & 0.1997 & 0.1079 & 0.1792 \\ 0.1187 & 0.0458 & 0.2114 & 0.0040 & 0.0527 & 0.1845 & 0.1692 & 0.2136 \\ 0.0045 & 0.0984 & 0.0024 & 0.1811 & 0.1630 & 0.1040 & 0.1692 & 0.2776 \end{bmatrix}$$

$$\text{estimate\_stayer} = \begin{bmatrix} 0.0556 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0526 & -0.0000 & -0.0000 \end{bmatrix}$$

$$estimate\_mover = \begin{bmatrix} 0.2605 & 0.1826 & 0.1796 & 0.0150 & 0.0629 & 0.1856 & 0.0419 & 0.0719 \\ 0.0556 & 0.1894 & 0.2020 & 0.0480 & 0.2652 & 0.0076 & 0.0429 & 0.1894 \\ 0.1107 & 0.1510 & 0.2047 & 0.0570 & 0.0604 & 0.1477 & 0.0470 & 0.2215 \\ 0.0969 & 0.1542 & 0.1542 & 0.2599 & 0.0749 & 0.0485 & 0.1101 & 0.1013 \\ 0.1893 & 0.1840 & 0.0053 & 0.0373 & 0.2453 & 0.1787 & 0.0533 & 0.1067 \\ 0.1821 & 0.1373 & 0.0476 & 0.0420 & 0.0812 & 0.1849 & 0.1261 & 0.1989 \\ 0.0777 & 0.0495 & 0.1802 & 0.0035 & 0.0530 & 0.1661 & 0.2297 & 0.2403 \\ 0.0021 & 0.1031 & 0.0021 & 0.1918 & 0.1794 & 0.0990 & 0.1526 & 0.2701 \end{bmatrix}$$

The above matrices were obtained by using the Maximum Likelihood method, when we get the above matrix, we check the difference between the two matrices, and which is listed as followed:

$$Difference=\begin{bmatrix} 0.0125 & 0.0280 & 0.0035 & 0.0112 & 0.0209 & 0.0279 & 0.0154 & 0.0006 \\ 0.0018 & 0.0305 & 0.0114 & 0.0008 & 0.0483 & 0.0030 & 0.0012 & 0.0108 \\ 0.0234 & 0.0150 & 0.0036 & 0.0131 & 0.0426 & 0.0029 & 0.0042 & 0.0328 \\ 0.0061 & 0.0061 & 0.0129 & 0.0536 & 0.0034 & 0.0225 & 0.0174 & 0.0096 \\ 0.0153 & 0.0040 & 0.0060 & 0.0158 & 0.0176 & 0.0162 & 0.0058 & 0.0097 \\ 0.0303 & 0.0098 & 0.0227 & 0.0024 & 0.0234 & 0.0148 & 0.0182 & 0.0197 \\ 0.0410 & 0.0036 & 0.0312 & 0.0004 & 0.0003 & 0.0184 & 0.0604 & 0.0266 \\ 0.0024 & 0.0047 & 0.0003 & 0.0107 & 0.0164 & 0.0050 & 0.0166 & 0.0075 \end{bmatrix}$$

According to the above matrices, we find the maximum difference is 0.0604 which located in the 7th row and 7th column, and the minimum difference is 3.1031e-004 which is located in the 8th row and the 2nd column, and also in the 5th row and in 7th column. Of course, we can use the meaning method, that is to say, we find the largest and the least number. Then, we divide them by two after pulsing them together. At the same time, we can use another method, that is the sum (sum (Difference)) divided by 8 times 8. We find the relative error which is defined as following:

$$relative\ \ error=\frac{m_{true}-m_{estimate}}{m_{true}}$$

According to the above formula, we can get the stayer and the mover relative error is respectively 0.0052 and 1.3010e-018.

According to the above example, we can list a summarized table as following:

The number of the sequence equals to 100 and rating equals to 8.

The number of sequence	The length of the sequence	Average relative error		
		Stayer	Mover	Initial probabilities
100	30	0.7854	0.0145	0.2297
100	50	1.0329	0.0116	0.2621
100	100	0.7847	0.0069	0.1437
100	200	0.6430	0.0052	0.1428
100	500	0.4695	0.0038	0.1305

According to the analysis of the above table, obviously, we find that the longer of the sequence, the smaller of the difference refers to the Mover, the Stayer and the initial of the probability. In other words, if the number of sequence is fixed, the longer of the length of the sequence, the smoother of the Mover-Stayer model, that is to say, the difference becomes less. At the same time, the calculated "average relative error" in the table stands for the sum of the relative errors divided by the corresponding number. For example, the average relative error of mover is 0.0145 which can be obtained by the code of the program that is

sum(sum(abs(mover-estimate_mover)/mover))/64.

In the table, the number of sequences or realizations is controlled to be 100 to find the effects of increasing the length of the sequence on the accuracy of the estimates. Here, the number of the sequence can stands for 100 companies, and the length of the sequence can be represented with the histories, which we can call the years. So in the analysis of the above tables, the number can reflect the *n* sequences of bonds ratings of length *J*, and we assume that there are *n* bonds and the credit ratings of those bonds are recorded for a maximum of 30 years. We think that the credit follows the Mover-Stayer Model.

In order to give a full assessment of the parameter estimates, the length of the sequences increases and the results of the difference are shown in the table below. The length of the sequence equals to 100 and rating equals to 8.

The number of sequence	The length of the sequence	Average relative error		
		Mover	Stayer	Initial probabilities
30	100	0.1219	0.4326	0.2445
50	100	0.0806	0.5265	0.1587
100	100	0.0661	0.2958	0.2722
200	100	0.0441	0.4324	0.1392
500	100	0.0316	0.3331	0.1244

The results above indicate that when the sequence length is kept at 100, by increasing the number of sequence does improve the estimate. When the number of sequences increases to 500, the parameter estimates show that the model is good enough to predict the estimators. At the same time, we can observe the changes by changing the range of the rating in a similar way.

3.1 Numerical the prediction of the results of the of the credit ratings

In the above part, we can list the difference or errors; we do not think that it is enough. At the same time, we want to text the correct rate of the Mover-Stayer Model. From the analysis of the previous part, we have known that the estimates are so accurate that the error is very small, which is the number of the sequence equals to 500 and the length of the sequence equals to 100. Next, we will check one more parameter according to the Mover-Stayer model to see how well the Mover-Stayer Model and calculate the correct rate of the estimators compared with the true model.

We will adopt the Mover-Stayer model to estimate the one more parameter. Firstly, the length of the sequence is to be fixed, and we will change the number of the sequence to observe the correct rate, and calculate the percentage of the correctly. The number of the sequence is 300, 500 and 800 respectively, which are listed as following:

The length of the sequence equals to 300.

The number of sequences	The number of totally correct predicted ratings	Percentage of the correct
300	137	45.67%
500	281	56.36%
800	469	58.63%

According to the above table, it seems that the Mover-Stayer model does not have a high percentage of the correct. However, the tables show an increasing trend that is the longer of the number of the sequence, the higher of the percentage of the correctness. But we think that the model is good enough in the practice.

4. Summary

In the dissertation, firstly, we build the Mover-Stayer model which is the likelihood function of n independent realizations, and the parameter estimates that maximization of the likelihood function are derived. And at the same time, the hypothetical data that follows a predefined Mover-Stayer model are generated to check the accuracy of the estimates. When the number of sequences is large, the result from these data shows a satisfactory precision in using those formula derived in the dissertation to estimate the parameters of the Mover-Stayer model. Finally, we make some competitions according to the results of the program, and the results of the number illustrate the strength of the true model studied.

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