Brand Selection and Its Matrix Structure

—Expansion of its Block Matrix to the Third Order Lag—

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Abstract

Focusing that consumers' are apt to buy superior brand when they are accustomed or bored to use current brand, new analysis method is introduced.

Before buying data and after buying data is stated using liner model. When above stated events occur, transition matrix becomes upper triangular matrix. In this paper, equation using transition matrix stated by the Block Matrix is expanded to the third order lag and the method is newly re-built. These are confirmed by numerical examples. S-step forecasting model is also introduced.

This approach makes it possible to identify brand position in the market and it can be utilized for building useful and effective marketing plan.

Keywords: Brand selection, Matrix structure, Brand position, Third order lag

1. Introduction

It is often observed that consumers select upper class brand when they buy next time after they are bored to use current brand.

Suppose that former buying data and current buying data are gathered. Also suppose that upper brand is located upper in the variable array. Then transition matrix becomes upper triangular matrix under the supposition that former buying variables are set input and current buying variables are set output. If the top brand were selected from lower brand skipping intermediate brands, corresponding part in upper triangular matrix would be 0. These are verified in numerical examples with simple models.

If transition matrix is identified, s-step forecasting can be executed. Generalized forecasting matrix components' equations are introduced. Unless planners for products notice its brand position whether it is upper or lower than other products, matrix structure makes it possible to identify those by calculating consumers' activities for brand selection. Thus, this proposed approach makes it effective to execute marketing plan and/or establish new brand.

Quantitative analysis concerning brand selection has been executed by Yamanaka (Yamanaka,H., 1982), Takahashi et al. (Takahashi,Y., T.Takahashi, 2002). Yamanaka(Yamanaka,H., 1982) examined purchasing process by Markov Transition Probability with the input of advertising expense. Takahashi et al. (Takahashi,Y., T.Takahashi, 2002) made analysis by the Brand Selection Probability model using logistics distribution.

In Takeyasu et al. (2008, 2011), matrix structure was analyzed for the case brand selection was executed toward upper class. In this paper, equation using transition matrix stated by the Block matrix is extended to the third order lag and the method is newly re-built. Such research as this cannot be found as long as searched.

Hereinafter, matrix structure is clarified for the selection of brand in section 2. Block matrix structure is analyzed when brands are handled in group and s -step forecasting is formulated in section 3. Expansion of the model to the third order lag is executed in section 4. Numerical calculation is executed in section 5. Application of this method is extended in section 6.

2. Brand Selection and its Matrix Structure

(1) Upper shift of Brand selection

Now, suppose that x is the most upper class brand, y is the second upper class brand, and z is the lowest class brand.

Consumer's behavior of selecting brand might be $z \rightarrow y$, $y \rightarrow x$, $z \rightarrow x$ etc. $x \rightarrow z$ might be few.

Suppose that x is current buying variable, and x_b is previous buying variable. Shift to x is executed from x_b, y_b, or z_b.

Therefore, x is stated in the following equation. a_{ij} represents transition probability from j-th to i-th brand.

$$x = a_{11}x_b + a_{12}y_b + a_{13}z_b$$

Similarly,

$$y = a_{22}y_b + a_{23}z_b$$

and

$$z = a_{33}z_b$$

These are re-written as follows.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} \tag{1}$$

Set

$$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}, \qquad \mathbf{X_b} = \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

then, $\mathbf{X}$ is represented as follows.

$$\mathbf{X} = \mathbf{A}\mathbf{X_b} \tag{2}$$

Here,

$$\mathbf{X} \in \mathbf{R}^3, \mathbf{A} \in \mathbf{R}^{3\times3}, \mathbf{X_b} \in \mathbf{R}^3$$

$\mathbf{A}$ is an upper triangular matrix.

To examine this, generating following data, which are all consisted by the data in which transition is made from lower brand to upper brand,

$$\mathbf{X^i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \cdots \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{3}$$

$$\mathbf{X_b^i} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \cdots \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4}$$

$$i = 1 \quad , \qquad 2 \qquad \cdots \qquad N$$

parameter can be estimated using least square method.

Suppose

$$\mathbf{X}^i = \mathbf{A}\mathbf{X_b}^i + \boldsymbol{\varepsilon}^i \tag{5}$$

where $\varepsilon^i = \begin{pmatrix} \varepsilon_1^i \\ \varepsilon_2^i \\ \varepsilon_3^i \end{pmatrix}$ $\quad i = 1, 2, \cdots, N$

and minimize following J

$$J = \sum_{i=1}^{N} \boldsymbol{\varepsilon}^{iT} \boldsymbol{\varepsilon}^i \to Min \tag{6}$$

$\hat{\mathbf{A}}$ which is an estimated value of $\mathbf{A}$ is obtained as follows.

$$\hat{\mathbf{A}} = \left(\sum_{i=1}^{N} \mathbf{X}^i \mathbf{X}_{\mathbf{b}}^{iT} \right) \left(\sum_{i=1}^{N} \mathbf{X}_{\mathbf{b}}^i \mathbf{X}_{\mathbf{b}}^{iT} \right)^{-1} \tag{7}$$

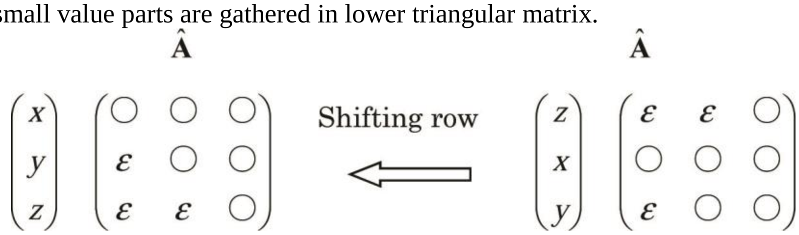
In the data group which are all consisted by the data in which transition is made from lower brand to upper brand, estimated value $\hat{\mathbf{A}}$ should be upper triangular matrix.
If following data which shift to lower brand are added only a few in equation (3) and (4),

$$\mathbf{X}^i = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{X}_{\mathbf{b}}^i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\hat{\mathbf{A}}$ would contain minute items in the lower part triangle.

(2) Sorting brand ranking by re-arranging row

In a general data, variables may not be in order as x, y, z. In that case, large and small value lie scattered in $\hat{\mathbf{A}}$. But re-arranging this, we can set in order by shifting row. The large value parts are gathered in upper triangular matrix, and the small value parts are gathered in lower triangular matrix.



(8)

(3) Matrix structure under the case skipping intermediate class brand is skipped

It is often observed that some consumers select the most upper class brand from the most lower class brand and skip selecting the intermediate class brand.

We suppose v, w, x, y, z brands (suppose they are laid from upper position to lower position as $v > w > x > y > z$).

In the above case, selection shifts would be

$$v \leftarrow z$$
$$v \leftarrow y$$

Suppose they do not shift to y, x, w from z, to x, w from y, and to w from x, then Matrix structure would be as follows.

$$\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix} \begin{pmatrix} v_b \\ w_b \\ x_b \\ y_b \\ z_b \end{pmatrix} \tag{9}$$

We confirm this by numerical example in section 4.

3. Block Matrix Structure in Brand Gourps and S-Step Forecasting

Next, we examine the case in brand groups. Matrices are composed by Block Matrix.

(1) Brand shift group — in the case of two groups

Suppose brand selection shifts from Corolla class to MarkⅡclass in car. In this case, it does not matter which company's car they choose. Thus, selection of cars are executed in a group and brand shift is considered to be done from group to group. Suppose brand groups at time n are as follows.

$\mathbf{X}$ consists of p varieties of goods, and $\mathbf{Y}$ consists of q varieties of goods.

$$\mathbf{X_n} = \begin{pmatrix} x_1^n \\ x_2^n \\ \vdots \\ x_p^n \end{pmatrix}, \qquad \mathbf{Y_n} = \begin{pmatrix} y_1^n \\ y_2^n \\ \vdots \\ y_q^n \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}, & \mathbf{A_{12}} \\ \mathbf{0}, & \mathbf{A_{22}} \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-1}} \\ \mathbf{Y_{n-1}} \end{pmatrix} \tag{10}$$

Here,

$$\mathbf{X_n} \in \mathbf{R}^p \ (n = 1,2,\cdots), \quad \mathbf{Y_n} \in \mathbf{R}^q \ (n = 1,2,\cdots), \quad \mathbf{A_{11}} \in \mathbf{R}^{p\times p}, \quad \mathbf{A_{12}} \in \mathbf{R}^{p\times q}, \quad \mathbf{A_{22}} \in \mathbf{R}^{q\times q}$$

Make one more step of shift, then we obtain following equation.

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}^2, & \mathbf{A_{11}}\mathbf{A_{12}} + \mathbf{A_{12}}\mathbf{A_{22}} \\ \mathbf{0}, & \mathbf{A_{22}}^2 \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-2}} \\ \mathbf{Y_{n-2}} \end{pmatrix} \tag{11}$$

Make one more step of shift again, then we obtain following equation.

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}^3, & \mathbf{A_{11}}^2\mathbf{A_{12}} + \mathbf{A_{11}}\mathbf{A_{12}}\mathbf{A_{22}} + \mathbf{A_{12}}\mathbf{A_{22}}^2 \\ \mathbf{0}, & \mathbf{A_{22}}^3 \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-3}} \\ \mathbf{Y_{n-3}} \end{pmatrix} \tag{12}$$

Similarly,

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}^4, & \mathbf{A_{11}}^3\mathbf{A_{12}} + \mathbf{A_{11}}^2\mathbf{A_{12}}\mathbf{A_{22}} + \mathbf{A_{11}}\mathbf{A_{12}}\mathbf{A_{22}}^2 + \mathbf{A_{12}}\mathbf{A_{22}}^3 \\ \mathbf{0}, & \mathbf{A_{22}}^4 \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-4}} \\ \mathbf{Y_{n-4}} \end{pmatrix} \tag{13}$$

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}^5, & \mathbf{A_{11}}^4\mathbf{A_{12}} + \mathbf{A_{11}}^3\mathbf{A_{12}}\mathbf{A_{22}} + \mathbf{A_{11}}^2\mathbf{A_{12}}\mathbf{A_{22}}^2 + \mathbf{A_{11}}\mathbf{A_{12}}\mathbf{A_{22}}^3 + \mathbf{A_{12}}\mathbf{A_{22}}^4 \\ \mathbf{0}, & \mathbf{A_{22}}^5 \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-5}} \\ \mathbf{Y_{n-5}} \end{pmatrix} \tag{14}$$

Finally, we get generalized equation for s -step shift as follows.

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}^s, & \mathbf{A_{11}}^{s-1}\mathbf{A_{12}} + \sum_{k=2}^{s-1} \mathbf{A_{11}}^{s-k}\mathbf{A_{12}}\mathbf{A_{22}}^{k-1} + \mathbf{A_{12}}\mathbf{A_{22}}^{s-1} \\ \mathbf{0}, & \mathbf{A_{22}}^s \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-s}} \\ \mathbf{Y_{n-s}} \end{pmatrix} \tag{15}$$

If we replace $n-s \to n, n \to n+s$ in equation (15), we can make s -step forecast.

(2) Brand shift group — in the case of three groups

Suppose brand selection is executed in the same group or to the upper group, and also suppose that brand position is $x > y > z$ (x is upper position). Then brand selection transition matrix would be expressed as

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \\ \mathbf{Z_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}, & \mathbf{A_{12}}, & \mathbf{A_{13}} \\ \mathbf{0}, & \mathbf{A_{22}}, & \mathbf{A_{23}} \\ \mathbf{0}, & \mathbf{0}, & \mathbf{A_{33}} \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-1}} \\ \mathbf{Y_{n-1}} \\ \mathbf{Z_{n-1}} \end{pmatrix} \tag{16}$$

Where

$$\mathbf{X_n} = \begin{pmatrix} x_1^n \\ x_2^n \\ \vdots \\ x_p^n \end{pmatrix}, \qquad \mathbf{Y_n} = \begin{pmatrix} y_1^n \\ y_2^n \\ \vdots \\ y_q^n \end{pmatrix}, \qquad \mathbf{Z_n} = \begin{pmatrix} z_1^n \\ z_2^n \\ \vdots \\ z_r^n \end{pmatrix}$$

Here,

$$\mathbf{X_n} \in \mathbf{R}^p \ (n = 1,2,\cdots), \quad \mathbf{Y_n} \in \mathbf{R}^q \ (n = 1,2,\cdots), \quad \mathbf{Z_n} \in \mathbf{R}^r \ (n = 1,2,\cdots), \quad \mathbf{A_{11}} \in R^{p\times p}, \quad \mathbf{A_{12}} \in R^{p\times q}$$

$$\mathbf{A_{13}} \in R^{p\times r}, \quad \mathbf{A_{22}} \in R^{q\times q}, \quad \mathbf{A_{23}} \in R^{q\times r}, \quad \mathbf{A_{33}} \in R^{r\times r}$$

These are re-stated as

$$\mathbf{W_n} = \mathbf{A}\mathbf{W_{n-1}} \tag{17}$$

where,

$$\mathbf{W_n} = \begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \\ \mathbf{Z_n} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} \mathbf{A_{11}}, & \mathbf{A_{12}}, & \mathbf{A_{13}} \\ \mathbf{0}, & \mathbf{A_{22}}, & \mathbf{A_{23}} \\ \mathbf{0}, & \mathbf{0}, & \mathbf{A_{33}} \end{pmatrix}, \qquad \mathbf{W_{n-1}} = \begin{pmatrix} \mathbf{X_{n-1}} \\ \mathbf{Y_{n-1}} \\ \mathbf{Z_{n-1}} \end{pmatrix}$$

Hereinafter, we shift steps as is done in previous section.
In the general description, we state as

$$\mathbf{W_n} = \mathbf{A^{(s)}}\mathbf{W_{n-s}} \tag{18}$$

Here,

$$\mathbf{A^{(s)}} = \begin{pmatrix} \mathbf{A_{11}}^{(s)}, & \mathbf{A_{12}}^{(s)}, & \mathbf{A_{13}}^{(s)} \\ \mathbf{0}, & \mathbf{A_{22}}^{(s)}, & \mathbf{A_{23}}^{(s)} \\ \mathbf{0}, & \mathbf{0}, & \mathbf{A_{33}}^{(s)} \end{pmatrix}, \qquad \mathbf{W_{n-s}} = \begin{pmatrix} \mathbf{X_{n-s}} \\ \mathbf{Y_{n-s}} \\ \mathbf{Z_{n-s}} \end{pmatrix}$$

From definition,

$$\mathbf{A^{(1)}} = \mathbf{A} \tag{19}$$

In the case $s = 2$, we obtain

$$\begin{aligned} \mathbf{A}^{(2)} &= \begin{pmatrix} \mathbf{A_{11}}, & \mathbf{A_{12}}, & \mathbf{A_{13}} \\ \mathbf{0}, & \mathbf{A_{22}}, & \mathbf{A_{23}} \\ \mathbf{0}, & \mathbf{0}, & \mathbf{A_{33}} \end{pmatrix} \begin{pmatrix} \mathbf{A_{11}}, & \mathbf{A_{12}}, & \mathbf{A_{13}} \\ \mathbf{0}, & \mathbf{A_{22}}, & \mathbf{A_{23}} \\ \mathbf{0}, & \mathbf{0}, & \mathbf{A_{33}} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{A}_{11}^2, & \mathbf{A_{11}A_{12}} + \mathbf{A_{12}A_{22}}, & \mathbf{A_{11}A_{13}} + \mathbf{A_{12}A_{23}} + \mathbf{A_{13}A_{33}} \\ \mathbf{0}, & \mathbf{A}_{22}^2, & \mathbf{A_{22}A_{23}} + \mathbf{A_{23}A_{33}} \\ \mathbf{0}, & \mathbf{0}, & \mathbf{A}_{33}^2 \end{pmatrix} \end{aligned} \tag{20}$$

Next, in the case $s = 3$, we obtain

$$\mathbf{A}^{(3)} = \begin{pmatrix} \mathbf{A}_{11}^3, & \mathbf{A}_{11}^2\mathbf{A}_{12} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22} + \mathbf{A}_{12}\mathbf{A}_{22}^2, & \mathbf{A}_{11}^2\mathbf{A}_{13} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{23} + \mathbf{A}_{11}\mathbf{A}_{13}\mathbf{A}_{33} + \mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23} + \mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{13}\mathbf{A}_{33}^2 \\ \mathbf{0}, & \mathbf{A}_{22}^3, & \mathbf{A}_{22}^2\mathbf{A}_{23} + \mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{23}\mathbf{A}_{33}^2 \\ \mathbf{0}, & \mathbf{0}, & \mathbf{A}_{33}^3 \end{pmatrix} \tag{21}$$

In the case $s = 4$, equations become wide-spread, so we express each Block Matrix as follows.

$$\left.\begin{aligned}
&\mathbf{A}_{11}^{(4)} = \mathbf{A}_{11}^{4} \\
&\mathbf{A}_{12}^{(4)} = \mathbf{A}_{11}^{3}\mathbf{A}_{12} + \mathbf{A}_{11}^{2}\mathbf{A}_{12}\mathbf{A}_{22} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^{2} + \mathbf{A}_{12}\mathbf{A}_{22}^{3} \\
&\mathbf{A}_{13}^{(4)} = \mathbf{A}_{11}^{3}\mathbf{A}_{13} + \mathbf{A}_{11}^{2}\mathbf{A}_{12}\mathbf{A}_{23} + \mathbf{A}_{11}^{2}\mathbf{A}_{13}\mathbf{A}_{33} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{11}\mathbf{A}_{13}\mathbf{A}_{33}^{2} \\
&\qquad + \mathbf{A}_{12}\mathbf{A}_{22}^{2}\mathbf{A}_{23} + \mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}^{2} + \mathbf{A}_{13}\mathbf{A}_{33}^{3} \\
&\mathbf{A}_{22}^{(4)} = \mathbf{A}_{22}^{4} \\
&\mathbf{A}_{23}^{(4)} = \mathbf{A}_{22}^{3}\mathbf{A}_{23} + \mathbf{A}_{22}^{2}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33}^{2} + \mathbf{A}_{23}\mathbf{A}_{33}^{3} \\
&\mathbf{A}_{33}^{(4)} = \mathbf{A}_{33}^{4}
\end{aligned}\right\} \qquad (22)$$

In the case $s = 5$, we obtain the following equations similarly.

$$\left.\begin{aligned}
&\mathbf{A}_{11}^{(5)} = \mathbf{A}_{11}^{5} \\
&\mathbf{A}_{12}^{(5)} = \mathbf{A}_{11}^{4}\mathbf{A}_{12} + \mathbf{A}_{11}^{3}\mathbf{A}_{12}\mathbf{A}_{22} + \mathbf{A}_{11}^{2}\mathbf{A}_{12}\mathbf{A}_{22}^{2} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^{3} + \mathbf{A}_{12}\mathbf{A}_{22}^{4} \\
&\mathbf{A}_{13}^{(5)} = \mathbf{A}_{11}^{4}\mathbf{A}_{13} + \mathbf{A}_{11}^{3}\mathbf{A}_{12}\mathbf{A}_{23} + \mathbf{A}_{11}^{3}\mathbf{A}_{13}\mathbf{A}_{33} + \mathbf{A}_{11}^{2}\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23} + \mathbf{A}_{11}^{2}\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{11}^{2}\mathbf{A}_{13}\mathbf{A}_{33}^{2} \\
&\qquad + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^{2}\mathbf{A}_{23} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}^{2} + \mathbf{A}_{11}\mathbf{A}_{13}\mathbf{A}_{33}^{3} \\
&\qquad + \mathbf{A}_{12}\mathbf{A}_{22}^{3}\mathbf{A}_{23} + \mathbf{A}_{12}\mathbf{A}_{22}^{2}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33}^{2} + \mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}^{3} + \mathbf{A}_{13}\mathbf{A}_{33}^{4} \\
&\mathbf{A}_{22}^{(5)} = \mathbf{A}_{22}^{5} \\
&\mathbf{A}_{23}^{(5)} = \mathbf{A}_{22}^{4}\mathbf{A}_{23} + \mathbf{A}_{22}^{3}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{22}^{2}\mathbf{A}_{23}\mathbf{A}_{33}^{2} + \mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33}^{3} + \mathbf{A}_{23}\mathbf{A}_{33}^{4} \\
&\mathbf{A}_{33}^{(5)} = \mathbf{A}_{33}^{5}
\end{aligned}\right\} \qquad (23)$$

In the case $s = 6$, we obtain

$$\left.\begin{aligned}
&\mathbf{A}_{11}^{(6)} = \mathbf{A}_{11}^{6} \\
&\mathbf{A}_{12}^{(6)} = \mathbf{A}_{11}^{5}\mathbf{A}_{12} + \mathbf{A}_{11}^{4}\mathbf{A}_{12}\mathbf{A}_{22} + \mathbf{A}_{11}^{3}\mathbf{A}_{12}\mathbf{A}_{22}^{2} + \mathbf{A}_{11}^{2}\mathbf{A}_{12}\mathbf{A}_{22}^{3} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^{4} + \mathbf{A}_{12}\mathbf{A}_{22}^{5} \\
&\mathbf{A}_{13}^{(6)} = \mathbf{A}_{11}^{5}\mathbf{A}_{13} + \mathbf{A}_{11}^{4}\mathbf{A}_{12}\mathbf{A}_{23} + \mathbf{A}_{11}^{4}\mathbf{A}_{13}\mathbf{A}_{33} + \mathbf{A}_{11}^{3}\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23} + \mathbf{A}_{11}^{3}\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{11}^{3}\mathbf{A}_{13}\mathbf{A}_{33}^{2} \\
&\qquad + \mathbf{A}_{11}^{2}\mathbf{A}_{12}\mathbf{A}_{22}^{2}\mathbf{A}_{23} + \mathbf{A}_{11}^{2}\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{11}^{2}\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}^{2} + \mathbf{A}_{11}^{2}\mathbf{A}_{13}\mathbf{A}_{33}^{3} \\
&\qquad + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^{3}\mathbf{A}_{23} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^{2}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33}^{2} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}^{3} + \mathbf{A}_{11}\mathbf{A}_{13}\mathbf{A}_{33}^{4} \\
&\qquad + \mathbf{A}_{12}\mathbf{A}_{22}^{4}\mathbf{A}_{23} + \mathbf{A}_{12}\mathbf{A}_{22}^{3}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{12}\mathbf{A}_{22}^{2}\mathbf{A}_{23}\mathbf{A}_{33}^{2} + \mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33}^{3} + \mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}^{4} + \mathbf{A}_{13}\mathbf{A}_{33}^{5}
\end{aligned}\right\} \qquad (24)$$

We get generalized equations for s-step shift as follows.

$$\left.\begin{aligned}
&\mathbf{A}_{11}^{(s)} = \mathbf{A}_{11}^{s} \\
&\mathbf{A}_{12}^{(s)} = \mathbf{A}_{11}^{s-1}\mathbf{A}_{12} + \sum_{k=2}^{s-1}\mathbf{A}_{11}^{s-k}\mathbf{A}_{12}\mathbf{A}_{22}^{k-1} + \mathbf{A}_{12}\mathbf{A}_{22}^{s-1} \\
&\mathbf{A}_{13}^{(s)} = \mathbf{A}_{11}^{s-1}\mathbf{A}_{13} + \mathbf{A}_{11}^{s-2}\left(\sum_{k=1}^{2}\mathbf{A}_{1(\mathbf{k}+1)}\mathbf{A}_{(\mathbf{k}+1)3}\right) + \sum_{j=1}^{s-3}\left[\mathbf{A}_{11}^{S-2-j}\left\{\mathbf{A}_{12}\left(\sum_{k=1}^{j+1}\mathbf{A}_{22}^{j+1-k}\mathbf{A}_{23}\mathbf{A}_{33}^{k-1}\right) + \mathbf{A}_{13}\mathbf{A}_{33}^{j+1}\right\}\right] \\
&\mathbf{A}_{22}^{(s)} = \mathbf{A}_{22}^{s} \\
&\mathbf{A}_{23}^{(s)} = \sum_{K-1}^{s}\mathbf{A}_{22}^{s-k}\mathbf{A}_{23}\mathbf{A}_{33}^{k-1} \\
&\mathbf{A}_{33}^{(s)} = \mathbf{A}_{33}^{s}
\end{aligned}\right\} \qquad (25)$$

Expressing them in matrix, it follows.

$$\mathbf{A}^{(S)}=\begin{pmatrix} \mathbf{A}_{11}^{s}, & \mathbf{A}_{11}^{s-1}\mathbf{A}_{12}+\sum_{k=2}^{s-1}\mathbf{A}_{11}^{s-k}\mathbf{A}_{12}\mathbf{A}_{22}^{k-1}+\mathbf{A}_{12}\mathbf{A}_{22}^{s-1}, & \mathbf{A}_{11}^{s-1}\mathbf{A}_{13}+\mathbf{A}_{11}^{s-2}\left(\sum_{k=1}^{2}\mathbf{A}_{1(k+1)}\mathbf{A}_{(K+1)3}\right)+\sum_{j=1}^{s-3}\left[\mathbf{A}_{11}^{s-2-j}\left\{\mathbf{A}_{12}\left(\sum_{k=1}^{j+1}\mathbf{A}_{22}^{j+1-k}\mathbf{A}_{23}\mathbf{A}_{33}^{k-1}\right)+\mathbf{A}_{13}\mathbf{A}_{33}^{j+1}\right\}\right] \\ \mathbf{0}, & \mathbf{A}_{22}^{s}, & \sum_{k=1}^{s}\mathbf{A}_{22}^{s-k}\mathbf{A}_{23}\mathbf{A}_{33}^{k-1} \\ \mathbf{0}, & \mathbf{0}, & \mathbf{A}_{33}^{s} \end{pmatrix} \tag{26}$$

Generalizing them to m groups, they are expressed as

$$\begin{pmatrix} \mathbf{X}_{\mathbf{n}}^{(1)} \\ \mathbf{X}_{\mathbf{n}}^{(2)} \\ \vdots \\ \mathbf{X}_{\mathbf{n}}^{(m)} \end{pmatrix}=\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1m} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2m} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{m1} & \mathbf{A}_{m2} & \cdots & \mathbf{A}_{mm} \end{pmatrix}\begin{pmatrix} \mathbf{X}_{\mathbf{n-1}}^{(1)} \\ \mathbf{X}_{\mathbf{n-1}}^{(2)} \\ \vdots \\ \mathbf{X}_{\mathbf{n-1}}^{(m)} \end{pmatrix} \tag{27}$$

$\mathbf{X}_{\mathbf{n}}^{(1)}\in R^{k_1},\quad \mathbf{X}_{\mathbf{n}}^{(2)}\in R^{k_2},\ \cdots,\quad \mathbf{X}_{\mathbf{n}}^{(m)}\in R^{k_m},\quad \mathbf{A}_{\mathbf{ij}}\in R^{k_i\times k_j}\ (i=1,\cdots,m)(j=1,\cdots,m)$

4. Expansion to the Third Order Lag

Expansion of the above stated Block Matrix model to the third order lag is executed in the following method.

Here we take three groups case.

Generating Eq.(16) and Eq.(18),we state the model as follows. Here we set P=3.

$$\begin{pmatrix} \mathbf{X}_n \\ \mathbf{Y}_n \\ \mathbf{Z}_n \end{pmatrix}=\begin{pmatrix} \mathbf{A}, & \mathbf{B}, & \mathbf{C} \\ \mathbf{D}, & \mathbf{E}, & \mathbf{F} \\ \mathbf{G}, & \mathbf{H}, & \mathbf{J} \end{pmatrix}\begin{pmatrix} \mathbf{X}_{n-1} \\ \mathbf{Y}_{n-1} \\ \mathbf{Z}_{n-1} \end{pmatrix} \tag{28}$$

Where

$$\mathbf{X}_n=\begin{pmatrix} x_1^n \\ x_2^n \\ x_3^n \end{pmatrix},\ \mathbf{Y}_n=\begin{pmatrix} y_1^n \\ y_2^n \\ y_3^n \end{pmatrix},\ \mathbf{Z}_{\mathbf{n}}=\begin{pmatrix} z_1^n \\ z_2^n \\ z_3^n \end{pmatrix} \tag{29}$$

Here,

$\mathbf{X}_n\in\mathbf{R}^3(n=1,2,\cdots),\ \mathbf{Y}_n\in\mathbf{R}^3(n=1,2,\cdots),\ \mathbf{Z}_n\in\mathbf{R}^{\mathbf{3}}(n=1,2,\cdots),\ \{\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D},\mathbf{E},\mathbf{F},\mathbf{G},\mathbf{H},\mathbf{J}\}\in\mathbf{R}^{3\times3}$

These are re-stated as:

$$\mathbf{W}_n=\mathbf{P}\mathbf{W}_{n-1} \tag{30}$$

$$\mathbf{W}_n=\begin{pmatrix} \mathbf{X}_n \\ \mathbf{Y}_n \\ \mathbf{Z}_n \end{pmatrix} \tag{31}$$

$$\mathbf{P}=\begin{pmatrix} \mathbf{A}, & \mathbf{B}, & \mathbf{C} \\ \mathbf{D}, & \mathbf{E}, & \mathbf{F} \\ \mathbf{G}, & \mathbf{H}, & \mathbf{J} \end{pmatrix} \tag{32}$$

$$\mathbf{W}_{n-1}=\begin{pmatrix} \mathbf{X}_{n-1} \\ \mathbf{Y}_{n-1} \\ \mathbf{Z}_{n-1} \end{pmatrix} \tag{33}$$

If N amount of data exist, we can derive the following the equation similarly as Eq.(5),

$$\mathbf{W}_n^i = \mathbf{P}\mathbf{W}_{n-1}^i + \boldsymbol{\varepsilon}_n^i \left(i = 1,2,\cdots,N\right) \tag{34}$$

and

$$J_n = \sum_{i=1}^{N} \boldsymbol{\varepsilon}_n^{iT} \boldsymbol{\varepsilon}_n^i \rightarrow Min \tag{35}$$

$\hat{\mathbf{P}}$ which is an estimated value of $\mathbf{P}$ is obtained as follows.

$$\hat{\mathbf{P}} = \left(\sum_{i=1}^{N} \mathbf{W}_n^i \mathbf{W}_{n-1}^{iT} \right) \left(\sum_{i=1}^{N} \mathbf{W}_{n-1}^i \mathbf{W}_{n-1}^{iT} \right)^{-1} \tag{36}$$

Now, we expand Eq.(34) to the third order lag model as follows.

$$\mathbf{W}_n^i = \mathbf{P}_1 \mathbf{W}_{n-1}^i + \mathbf{P}_2 \mathbf{W}_{n-2}^i + \mathbf{P}_3 \mathbf{W}_{n-3}^i + \boldsymbol{\varepsilon}_n^i \tag{37}$$

Here

$$\mathbf{P}_1 = \begin{pmatrix} \mathbf{A}_1, & \mathbf{B}_1, & \mathbf{C}_1 \\ \mathbf{D}_1, & \mathbf{E}_1, & \mathbf{F}_1 \\ \mathbf{G}_1, & \mathbf{H}_1, & \mathbf{J}_1 \end{pmatrix}, \mathbf{P}_2 = \begin{pmatrix} \mathbf{A}_2, & \mathbf{B}_2, & \mathbf{C}_2 \\ \mathbf{D}_2, & \mathbf{E}_2, & \mathbf{F}_2 \\ \mathbf{G}_2, & \mathbf{H}_2, & \mathbf{J}_2 \end{pmatrix}, \mathbf{P}_3 = \begin{pmatrix} \mathbf{A}_3, & \mathbf{B}_3, & \mathbf{C}_3 \\ \mathbf{D}_3, & \mathbf{E}_3, & \mathbf{F}_3 \\ \mathbf{G}_3, & \mathbf{H}_3, & \mathbf{J}_3 \end{pmatrix} \tag{38}$$

It we set

$$\mathbf{P} = \left(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\right) \tag{39}$$

then $\hat{\mathbf{P}}$ can be estimated as follows.

$$\mathbf{P} = \left(\sum_{i=1}^{N} \mathbf{W}_t^i \begin{pmatrix} \mathbf{W}_{t-1}^i \\ \mathbf{W}_{t-2}^i \\ \mathbf{W}_{t-3}^i \end{pmatrix}^T \right) \left(\sum_{i=1}^{N} \begin{pmatrix} \mathbf{W}_{t-1}^i \\ \mathbf{W}_{t-2}^i \\ \mathbf{W}_{t-3}^i \end{pmatrix} \begin{pmatrix} \mathbf{W}_{t-1}^i \\ \mathbf{W}_{t-2}^i \\ \mathbf{W}_{t-3}^i \end{pmatrix}^T \right)^{-1} \tag{40}$$

We further develop this equation as follows.

$$\begin{aligned}
\mathbf{P} &= \left(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\right) \\
&= \begin{pmatrix} \mathbf{A}_1, & \mathbf{B}_1, & \mathbf{C}_1, & \mathbf{A}_2, & \mathbf{B}_2, & \mathbf{C}_2, & \mathbf{A}_3, & \mathbf{B}_3, & \mathbf{C}_3 \\ \mathbf{D}_1, & \mathbf{E}_1, & \mathbf{F}_1, & \mathbf{D}_2, & \mathbf{E}_2, & \mathbf{F}_2, & \mathbf{D}_3, & \mathbf{E}_3, & \mathbf{F}_3 \\ \mathbf{G}_1, & \mathbf{H}_1, & \mathbf{J}_1, & \mathbf{G}_2, & \mathbf{H}_2, & \mathbf{J}_2, & \mathbf{G}_3, & \mathbf{H}_3, & \mathbf{J}_3 \end{pmatrix} \\
&= \left(\sum_{i=1}^{N} \mathbf{W}_t^i \mathbf{W}_{t-1}^{iT}, \sum_{i=1}^{N} \mathbf{W}_t^i \mathbf{W}_{t-2}^{iT}, \sum_{i=1}^{N} \mathbf{W}_t^i \mathbf{W}_{t-3}^{iT} \right) \begin{pmatrix} \sum_{i=1}^{N} \mathbf{W}_{t-1}^i \mathbf{W}_{t-1}^{iT}, \sum_{i=1}^{N} \mathbf{W}_{t-1}^i \mathbf{W}_{t-2}^{iT}, \sum_{i=1}^{N} \mathbf{W}_{t-1}^i \mathbf{W}_{t-3}^{iT} \\ \sum_{i=1}^{N} \mathbf{W}_{t-2}^i \mathbf{W}_{t-1}^{iT}, \sum_{i=1}^{N} \mathbf{W}_{t-2}^i \mathbf{W}_{t-2}^{iT}, \sum_{i=1}^{N} \mathbf{W}_{t-2}^i \mathbf{W}_{t-3}^{iT} \\ \sum_{i=1}^{N} \mathbf{W}_{t-3}^i \mathbf{W}_{t-1}^{iT}, \sum_{i=1}^{N} \mathbf{W}_{t-3}^i \mathbf{W}_{t-2}^{iT}, \sum_{i=1}^{N} \mathbf{W}_{t-3}^i \mathbf{W}_{t-3}^{iT} \end{pmatrix}^{-1} \\
&= \left(\sum_{i=1}^{N} \begin{pmatrix} \mathbf{x}_t^i \\ \mathbf{y}_t^i \\ \mathbf{z}_t^i \end{pmatrix} \left(\mathbf{x}_{t-1}^{iT}, \mathbf{y}_{t-1}^{iT}, \mathbf{z}_{t-1}^{iT}\right), \quad \sum_{i=1}^{N} \begin{pmatrix} \mathbf{x}_t^i \\ \mathbf{y}_t^i \\ \mathbf{z}_t^i \end{pmatrix} \left(\mathbf{x}_{t-2}^{iT}, \mathbf{y}_{t-2}^{iT}, \mathbf{z}_{t-2}^{iT}\right), \quad \sum_{i=1}^{N} \begin{pmatrix} \mathbf{x}_t^i \\ \mathbf{y}_t^i \\ \mathbf{z}_t^i \end{pmatrix} \left(\mathbf{x}_{t-3}^{iT}, \mathbf{y}_{t-3}^{iT}, \mathbf{z}_{t-3}^{iT}\right) \right)
\end{aligned}$$

$$\times\begin{pmatrix} \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}^{i}_{t-1}\\ \mathbf{y}^{i}_{t-1}\\ \mathbf{z}^{i}_{t-1}\end{pmatrix}\left(\mathbf{x}^{iT}_{t-1},\mathbf{y}^{iT}_{t-1},\mathbf{z}^{iT}_{t-1}\right), & \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}^{i}_{t-1}\\ \mathbf{y}^{i}_{t-1}\\ \mathbf{z}^{i}_{t-1}\end{pmatrix}\left(\mathbf{x}^{iT}_{t-2},\mathbf{y}^{iT}_{t-2},\mathbf{z}^{iT}_{t-2}\right), & \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}^{i}_{t-1}\\ \mathbf{y}^{i}_{t-1}\\ \mathbf{z}^{i}_{t-1}\end{pmatrix}\left(\mathbf{x}^{iT}_{t-3},\mathbf{y}^{iT}_{t-3},\mathbf{z}^{iT}_{t-3}\right) \\ \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}^{i}_{t-2}\\ \mathbf{y}^{i}_{t-2}\\ \mathbf{z}^{i}_{t-2}\end{pmatrix}\left(\mathbf{x}^{iT}_{t-1},\mathbf{y}^{iT}_{t-1},\mathbf{z}^{iT}_{t-1}\right), & \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}^{i}_{t-2}\\ \mathbf{y}^{i}_{t-2}\\ \mathbf{z}^{i}_{t-2}\end{pmatrix}\left(\mathbf{x}^{iT}_{t-2},\mathbf{y}^{iT}_{t-2},\mathbf{z}^{iT}_{t-2}\right), & \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}^{i}_{t-2}\\ \mathbf{y}^{i}_{t-2}\\ \mathbf{z}^{i}_{t-2}\end{pmatrix}\left(\mathbf{x}^{iT}_{t-3},\mathbf{y}^{iT}_{t-3},\mathbf{z}^{iT}_{t-3}\right) \\ \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}^{i}_{t-3}\\ \mathbf{y}^{i}_{t-3}\\ \mathbf{z}^{i}_{t-3}\end{pmatrix}\left(\mathbf{x}^{iT}_{t-1},\mathbf{y}^{iT}_{t-1},\mathbf{z}^{iT}_{t-1}\right), & \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}^{i}_{t-3}\\ \mathbf{y}^{i}_{t-3}\\ \mathbf{z}^{i}_{t-3}\end{pmatrix}\left(\mathbf{x}^{iT}_{t-2},\mathbf{y}^{iT}_{t-2},\mathbf{z}^{iT}_{t-2}\right), & \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}^{i}_{t-3}\\ \mathbf{y}^{i}_{t-3}\\ \mathbf{z}^{i}_{t-3}\end{pmatrix}\left(\mathbf{x}^{iT}_{t-3},\mathbf{y}^{iT}_{t-3},\mathbf{z}^{iT}_{t-3}\right) \end{pmatrix}^{-1} \tag{41}$$

$$=\begin{pmatrix} \sum_{i=1}^{N}\mathbf{x}^{i}_{t}\mathbf{x}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t}\mathbf{y}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t}\mathbf{z}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t}\mathbf{x}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t}\mathbf{y}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t}\mathbf{z}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t}\mathbf{x}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t}\mathbf{y}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t}\mathbf{z}^{iT}_{t-3} \\ \sum_{i=1}^{N}\mathbf{y}^{i}_{t}\mathbf{x}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t}\mathbf{y}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t}\mathbf{z}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t}\mathbf{x}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t}\mathbf{y}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t}\mathbf{z}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t}\mathbf{x}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t}\mathbf{y}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t}\mathbf{z}^{iT}_{t-3} \\ \sum_{i=1}^{N}\mathbf{z}^{i}_{t}\mathbf{x}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t}\mathbf{y}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t}\mathbf{z}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t}\mathbf{x}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t}\mathbf{y}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t}\mathbf{z}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t}\mathbf{x}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t}\mathbf{y}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t}\mathbf{z}^{iT}_{t-3} \end{pmatrix}$$

$$=\left(\begin{array}{ccc|ccc|ccc} \sum_{i=1}^{N}\mathbf{x}^{i}_{t-1}\mathbf{x}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-1}\mathbf{y}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-1}\mathbf{z}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-1}\mathbf{x}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-1}\mathbf{y}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-1}\mathbf{z}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-1}\mathbf{x}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-1}\mathbf{y}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-1}\mathbf{z}^{iT}_{t-3} \\ \sum_{i=1}^{N}\mathbf{y}^{i}_{t-1}\mathbf{x}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-1}\mathbf{y}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-1}\mathbf{z}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-1}\mathbf{x}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-1}\mathbf{y}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-1}\mathbf{z}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-1}\mathbf{x}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-1}\mathbf{y}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-1}\mathbf{z}^{iT}_{t-3} \\ \sum_{i=1}^{N}\mathbf{z}^{i}_{t-1}\mathbf{x}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-1}\mathbf{y}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-1}\mathbf{z}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-1}\mathbf{x}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-1}\mathbf{y}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-1}\mathbf{z}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-1}\mathbf{x}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-1}\mathbf{y}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-1}\mathbf{z}^{iT}_{t-3} \\ \hline \sum_{i=1}^{N}\mathbf{x}^{i}_{t-2}\mathbf{x}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-2}\mathbf{y}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-2}\mathbf{z}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-2}\mathbf{x}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-2}\mathbf{y}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-2}\mathbf{z}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-2}\mathbf{x}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-2}\mathbf{y}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-2}\mathbf{z}^{iT}_{t-3} \\ \sum_{i=1}^{N}\mathbf{y}^{i}_{t-2}\mathbf{x}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-2}\mathbf{y}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-2}\mathbf{z}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-2}\mathbf{x}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-2}\mathbf{y}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-2}\mathbf{z}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-2}\mathbf{x}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-2}\mathbf{y}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-2}\mathbf{z}^{iT}_{t-3} \\ \sum_{i=1}^{N}\mathbf{z}^{i}_{t-2}\mathbf{x}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-2}\mathbf{y}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-2}\mathbf{z}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-2}\mathbf{x}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-2}\mathbf{y}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-2}\mathbf{z}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-2}\mathbf{x}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-2}\mathbf{y}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-2}\mathbf{z}^{iT}_{t-3} \\ \hline \sum_{i=1}^{N}\mathbf{x}^{i}_{t-3}\mathbf{x}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-3}\mathbf{y}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-3}\mathbf{z}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-3}\mathbf{x}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-3}\mathbf{y}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-3}\mathbf{z}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-3}\mathbf{x}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-3}\mathbf{y}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{x}^{i}_{t-3}\mathbf{z}^{iT}_{t-3} \\ \sum_{i=1}^{N}\mathbf{y}^{i}_{t-3}\mathbf{x}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-3}\mathbf{y}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-3}\mathbf{z}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-3}\mathbf{x}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-3}\mathbf{y}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-3}\mathbf{z}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-3}\mathbf{x}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-3}\mathbf{y}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{y}^{i}_{t-3}\mathbf{z}^{iT}_{t-3} \\ \sum_{i=1}^{N}\mathbf{z}^{i}_{t-3}\mathbf{x}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-3}\mathbf{y}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-3}\mathbf{z}^{iT}_{t-1}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-3}\mathbf{x}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-3}\mathbf{y}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-3}\mathbf{z}^{iT}_{t-2}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-3}\mathbf{x}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-3}\mathbf{y}^{iT}_{t-3}, & \sum_{i=1}^{N}\mathbf{z}^{i}_{t-3}\mathbf{z}^{iT}_{t-3} \end{array}\right)$$

We set this as:

$$\mathbf{P}=\left(\mathbf{P}_{1},\mathbf{P}_{2},\mathbf{P}_{3}\right)$$

$$
\mathbf{P}=\left(\begin{array}{ccc:ccc:ccc}
\mathbf{K}_1 & \mathbf{K}_2 & \mathbf{K}_3 & \mathbf{L}_1 & \mathbf{L}_2 & \mathbf{L}_3 & \mathbf{M}_1 & \mathbf{M}_2 & \mathbf{M}_3\\
\mathbf{K}_4 & \mathbf{K}_5 & \mathbf{K}_6 & \mathbf{L}_4 & \mathbf{L}_5 & \mathbf{L}_6 & \mathbf{M}_4 & \mathbf{M}_5 & \mathbf{M}_6\\
\mathbf{K}_7 & \mathbf{K}_8 & \mathbf{K}_9 & \mathbf{L}_7 & \mathbf{L}_8 & \mathbf{L}_9 & \mathbf{M}_7 & \mathbf{M}_8 & \mathbf{M}_9
\end{array}\right)
$$

$$
\times\left(\begin{array}{ccc:ccc:ccc}
\mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 & \mathbf{Q}_1 & \mathbf{Q}_2 & \mathbf{Q}_3 & \mathbf{R}_1 & \mathbf{R}_2 & \mathbf{R}_3\\
\mathbf{N}_4 & \mathbf{N}_5 & \mathbf{N}_6 & \mathbf{Q}_4 & \mathbf{Q}_5 & \mathbf{Q}_6 & \mathbf{R}_4 & \mathbf{R}_5 & \mathbf{R}_6\\
\mathbf{N}_7 & \mathbf{N}_8 & \mathbf{N}_9 & \mathbf{Q}_7 & \mathbf{Q}_8 & \mathbf{Q}_9 & \mathbf{R}_7 & \mathbf{R}_8 & \mathbf{R}_9\\
\hdashline
\mathbf{S}_1 & \mathbf{S}_2 & \mathbf{S}_3 & \mathbf{T}_1 & \mathbf{T}_2 & \mathbf{T}_3 & \mathbf{U}_1 & \mathbf{U}_2 & \mathbf{U}_3\\
\mathbf{S}_4 & \mathbf{S}_5 & \mathbf{S}_6 & \mathbf{T}_4 & \mathbf{T}_5 & \mathbf{T}_6 & \mathbf{U}_4 & \mathbf{U}_5 & \mathbf{U}_6\\
\mathbf{S}_7 & \mathbf{S}_8 & \mathbf{S}_9 & \mathbf{T}_7 & \mathbf{T}_8 & \mathbf{T}_9 & \mathbf{U}_7 & \mathbf{U}_8 & \mathbf{U}_9\\
\hdashline
\mathbf{V}_1 & \mathbf{V}_2 & \mathbf{V}_3 & \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 & \boldsymbol{\alpha}_3 & \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 & \boldsymbol{\beta}_3\\
\mathbf{V}_4 & \mathbf{V}_5 & \mathbf{V}_6 & \boldsymbol{\alpha}_4 & \boldsymbol{\alpha}_5 & \boldsymbol{\alpha}_6 & \boldsymbol{\beta}_4 & \boldsymbol{\beta}_5 & \boldsymbol{\beta}_6\\
\mathbf{V}_7 & \mathbf{V}_8 & \mathbf{V}_9 & \boldsymbol{\alpha}_7 & \boldsymbol{\alpha}_8 & \boldsymbol{\alpha}_9 & \boldsymbol{\beta}_7 & \boldsymbol{\beta}_8 & \boldsymbol{\beta}_9
\end{array}\right)^{-1}
$$

Then when all consist of the same level shifts or the upper level shifts (suppose $\mathbf{X} > \mathbf{Y} > \mathbf{Z}$),

$\boldsymbol{K}_4, \boldsymbol{K}_7, \boldsymbol{K}_8, \boldsymbol{L}_4, \boldsymbol{L}_7, \boldsymbol{L}_8, \boldsymbol{M}_2, \boldsymbol{M}_7, \boldsymbol{M}_8, \boldsymbol{N}_2, \boldsymbol{N}_3, \boldsymbol{N}_6, \boldsymbol{T}_2, \boldsymbol{T}_3, \boldsymbol{T}_6, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3, \boldsymbol{\beta}_6, \boldsymbol{Q}_4, \boldsymbol{Q}_7, \boldsymbol{Q}_8, \boldsymbol{R}_4, \boldsymbol{R}_7, \boldsymbol{R}_8, \boldsymbol{U}_4, \boldsymbol{U}_7, \boldsymbol{U}_8$

are all 0.

As

$$
\begin{array}{llll}
\boldsymbol{N}_4 = \boldsymbol{N}_2^T, & \boldsymbol{N}_7 = \boldsymbol{N}_3^T, & \boldsymbol{N}_8 = \boldsymbol{N}_6^T, & \boldsymbol{S}_2 = \boldsymbol{Q}_4^T, \boldsymbol{S}_3 = \boldsymbol{Q}_7^T, \boldsymbol{S}_6 = \boldsymbol{Q}_8^T\\
\boldsymbol{T}_4 = \boldsymbol{T}_2^T, & \boldsymbol{T}_7 = \boldsymbol{T}_3^T, & \boldsymbol{T}_8 = \boldsymbol{T}_6^T, & \boldsymbol{V}_2 = \boldsymbol{R}_4^T, \boldsymbol{V}_3 = \boldsymbol{R}_7^T, \boldsymbol{V}_6 = \boldsymbol{R}_8^T\\
\boldsymbol{\beta}_4 = \boldsymbol{\beta}_2^T, & \boldsymbol{\beta}_7 = \boldsymbol{\beta}_3^T, & \boldsymbol{\beta}_8 = \boldsymbol{\beta}_6^T, & \boldsymbol{\alpha}_2 = \boldsymbol{U}_4^T, \boldsymbol{\alpha}_3 = \boldsymbol{U}_7^T, \boldsymbol{\alpha}_6 = \boldsymbol{U}_8^T
\end{array}
$$

therefore they are all 0.

$\boldsymbol{N}_1, \boldsymbol{N}_5, \boldsymbol{N}_9, \boldsymbol{T}_1, \boldsymbol{T}_5, \boldsymbol{T}_9, \boldsymbol{\beta}_1, \boldsymbol{\beta}_5, \boldsymbol{\beta}_9$ become diagonal Matrices.

Using a symbol "$*$" as a diagonal matrix, $\mathbf{P}$ becomes as follows by using the relation stated above.

$$
\mathbf{P}=\left(\begin{array}{ccc:ccc:ccc}
\mathbf{K}_1, & \mathbf{K}_2, & \mathbf{K}_3, & \mathbf{L}_1, & \mathbf{L}_2, & \mathbf{L}_3, & \mathbf{M}_1, & \mathbf{M}_2, & \mathbf{M}_3\\
\mathbf{0}, & \mathbf{K}_5, & \mathbf{K}_6, & \mathbf{0}, & \mathbf{L}_5, & \mathbf{L}_6, & \mathbf{0}, & \mathbf{M}_5, & \mathbf{M}_6\\
\mathbf{0}, & \mathbf{0}, & \mathbf{K}_9, & \mathbf{0}, & \mathbf{0}, & \mathbf{L}_9, & \mathbf{0}, & \mathbf{0}, & \mathbf{M}_9
\end{array}\right)
$$

$$
\times\left(\begin{array}{ccc:ccc:ccc}
*, & \mathbf{0}, & \mathbf{0}, & \mathbf{N}_1, & \mathbf{N}_2, & \mathbf{N}_3, & \mathbf{R}_1, & \mathbf{R}_2, & \mathbf{R}_3\\
\mathbf{0}, & *, & \mathbf{0}, & \mathbf{0}, & \mathbf{N}_5, & \mathbf{N}_6, & \mathbf{0}, & \mathbf{R}_5, & \mathbf{R}_6\\
\mathbf{0}, & \mathbf{0}, & *, & \mathbf{0}, & \mathbf{0}, & \mathbf{N}_9, & \mathbf{0}, & \mathbf{0}, & \mathbf{R}_9\\
\hdashline
\mathbf{S}_1, & \mathbf{0} & \mathbf{0}, & *, & \mathbf{0}, & \mathbf{0}, & \mathbf{U}_1 & \mathbf{U}_2 & \mathbf{U}_3\\
\mathbf{S}_4, & \mathbf{S}_5, & \mathbf{0}, & \mathbf{0}, & *, & \mathbf{0}, & \mathbf{0}, & \mathbf{U}_5 & \mathbf{U}_6\\
\mathbf{S}_7, & \mathbf{S}_8, & \mathbf{S}_9, & \mathbf{0}, & \mathbf{0}, & *, & \mathbf{0}, & \mathbf{0}, & \mathbf{U}_9\\
\hdashline
\mathbf{V}_1, & \mathbf{0}, & \mathbf{0}, & \boldsymbol{\alpha}_1, & \mathbf{0}, & \mathbf{0}, & *, & \mathbf{0}, & \mathbf{0},\\
\mathbf{V}_4, & \mathbf{V}_5, & \mathbf{0}, & \boldsymbol{\alpha}_4, & \boldsymbol{\alpha}_5, & \mathbf{0}, & \mathbf{0}, & *, & \mathbf{0},\\
\mathbf{V}_7, & \mathbf{V}_8, & \mathbf{V}_9, & \boldsymbol{\alpha}_7, & \boldsymbol{\alpha}_8, & \boldsymbol{\alpha}_9, & \mathbf{0}, & \mathbf{0}, & *
\end{array}\right)^{-1}
$$

5. Numerical Example

We consider the case that brand selection shifts to the same class or upper classes. As above-referenced, transition matrix must be an upper triangular matrix.

Suppose following events occur.

	X_{t-3} to X_{t-2}			X_{t-2} to X_{t-1}			X_{t-1} to X_t		
①	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
②	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
③	L_1	L_1	1 event	L_1	L_1	2 events	L_1	L_1	2 events
④	L_1	L_1	1 event	L_1	L_1	2 events	L_1	L_1	2 events
⑤	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
⑥	L_1	L_1	3 events	L_1	L_1	2 events	L_1	L_1	2 events
⑦	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
⑧	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
⑨	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
⑩	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
⑪	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
⑫	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
⑬	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
⑭	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
⑮	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
⑯	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
⑰	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
⑱	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
⑲	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
⑳	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
㉑	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
㉒	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
㉓	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
㉔	L_1	L_1	2 events	L_1	L_1	2 events	L_1	L_1	2 events
㉕	M_1	M_1	2 events	M_1	M_1	2 events	M_1	M_1	2 events
㉖	M_1	M_1	2 events	M_1	M_1	2 events	M_1	M_1	2 events
㉗	M_1	M_1	2 events	M_1	M_1	2 events	M_1	M_1	2 events
㉘	M_1	M_1	2 events	M_1	M_1	2 events	M_1	M_1	2 events
㉙	M_1	M_1	2 events	M_1	M_1	2 events	M_1	M_1	2 events
㉚	M_1	M_1	2 events	M_1	M_1	2 events	M_1	M_1	2 events
㉛	M_1	M_1	2 events	M_1	M_1	2 events	M_1	M_1	2 events
㉜	M_1	M_1	2 events	M_1	M_1	2 events	M_1	M_1	2 events
㉝	M_1	M_1	2 events	M_1	M_1	2 events	M_1	M_1	2 events
㉞	M_1	M_1	2 events	M_1	M_1	2 events	M_1	M_1	2 events
㉟	M_1	M_1	2 events	M_1	M_1	2 events	M_1	M_1	2 events
㊱	M_1	M_1	2 events	M_1	M_1	2 events	M_1	M_1	2 events

37	M_1	M_1	2 *events*	M_1	M_1	2 *events*	M_1	M_1	2 *events*
38	M_1	M_1	2 *events*	M_1	M_1	2 *events*	M_1	M_1	2 *events*
39	M_1	M_1	2 *events*	M_1	M_1	2 *events*	M_1	M_1	2 *events*
40	M_1	M_1	2 *events*	M_1	M_1	2 *events*	M_1	M_1	2 *events*
41	M_1	M_1	2 *events*	M_1	M_1	2 *events*	M_1	M_1	2 *events*
42	M_3	M_3	2 *events*	M_3	U_2	2 *events*			
43	M_3	M_3	2 *events*	M_3	U_3	2 *events*			
44	U_1	U_1	1 *event*	U_1	U_1	1 *event*			
45	U_1	U_1	1 *event*	U_1	U_2	1 *event*			
46	U_1	U_1	2 *events*	U_1	U_3	2 *events*			
47	U_2	U_2	1 *event*	U_2	U_2	1 *event*			
48	U_2	U_2	2 *events*	U_2	U_3	2 *events*			
49	U_2	U_2	2 *events*	U_2	U_1	2 *events*			
50	U_3	U_3	3 *events*	U_3	U_3	3 *events*			
51	U_3	U_3	2 *events*	U_3	U_2	2 *events*			
52	U_3	U_3	1 *event*	U_3	U_1	1 *event*			
53	M_3	M_3	1 *event*	M_3	M_2	1 *event*			
54	M_3	M_3	2 *events*	M_3	M_1	2 *events*			
55	M_3	M_2	1 *event*	M_2	M_2	1 *event*			
56	M_3	M_2	1 *event*	M_2	M_1	1 *event*			
57	M_2	M_2	2 *events*	M_2	M_1	2 *events*			
58	L_3	L_3	3 *events*	L_3	L_2	3 *events*			
59	L_3	L_2	2 *events*	L_2	L_1	2 *events*			
60	L_3	L_1	1 *event*	L_1	L_1	1 *event*			
61				L_1	L_1	2 *events*			
62				L_1	L_2	2 *events*			
63				L_1	L_3	1 *event*			
64				L_1	M_1	1 *event*			
65				L_1	M_2	3 *events*			
66				L_2	L_2	2 *events*			
67				L_2	L_3	1 *event*			
68				L_2	L_1	2 *events*			
69				L_2	M_1	1 *event*			
70				L_2	U_1	1 *event*			
71				L_1	U_2	3 *events*			

⑫	M_1	M_1	$2\ events$
	X_{t-1} to X_t		
⑬	L_3	L_3	$2\ events$
⑭	M_2	M_2	$1\ event$
⑮	M_3	M_3	$1\ event$
⑯	M_1	M_2	$3\ events$
⑰	M_1	U_1	$3\ events$
⑱	M_2	U_3	$2\ events$
⑲	M_3	U_2	$1\ event$
⑳	U_1	U_1	$1\ event$
㉑	U_2	U_2	$2\ events$
㉒	U_3	U_3	$2\ events$
㉓	U_1	U_2	$1\ event$
㉔	U_1	U_3	$2\ events$
㉕	U_3	U_2	$2\ events$
㉖	U_3	U_1	$1\ event$
㉗	U_2	U_1	$1\ event$
㉘	U_2	U_3	$2\ events$
㉙	L_3	L_1	$3\ events$
㉚	M_2	L_1	$1\ event$

Vector $\begin{pmatrix} \boldsymbol{X}_{t-2} \\ \boldsymbol{Y}_{t-2} \\ \boldsymbol{Z}_{t-2} \end{pmatrix}, \begin{pmatrix} \boldsymbol{X}_{t-1} \\ \boldsymbol{Y}_{t-1} \\ \boldsymbol{Z}_{t-1} \end{pmatrix}, \begin{pmatrix} \boldsymbol{X}_{t} \\ \boldsymbol{Y}_{t} \\ \boldsymbol{Z}_{t} \end{pmatrix}$ in these cases are expressed as follows. We show some of them as an example.

①
$$\begin{pmatrix} \mathbf{X}_{t-2} \\ \mathbf{Y}_{t-2} \\ \mathbf{Z}_{t-2} \end{pmatrix} = \overset{X_{t-2}}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}, \quad \begin{pmatrix} \mathbf{X}_{t-1} \\ \mathbf{Y}_{t-1} \\ \mathbf{Z}_{t-1} \end{pmatrix} = \overset{X_{t-1}}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}, \quad \begin{pmatrix} \mathbf{X}_{t} \\ \mathbf{Y}_{t} \\ \mathbf{Z}_{t} \end{pmatrix} = \overset{X_{t}}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}$$

② $$\begin{pmatrix} \mathbf{X}_{t-2} \\ \mathbf{Y}_{t-2} \\ \mathbf{Z}_{t-2} \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\1\\0\\0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{X}_{t-1} \\ \mathbf{Y}_{t-1} \\ \mathbf{Z}_{t-1} \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\1\\0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{X}_{t} \\ \mathbf{Y}_{t} \\ \mathbf{Z}_{t} \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\1\\0 \end{pmatrix}$$

③ $$\begin{pmatrix} \mathbf{X}_{t-2} \\ \mathbf{Y}_{t-2} \\ \mathbf{Z}_{t-2} \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\1\\0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{X}_{t-1} \\ \mathbf{Y}_{t-1} \\ \mathbf{Z}_{t-1} \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\1\\0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{X}_{t} \\ \mathbf{Y}_{t} \\ \mathbf{Z}_{t} \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\1 \end{pmatrix}$$

④ $$\begin{pmatrix} \mathbf{X}_{t-2} \\ \mathbf{Y}_{t-2} \\ \mathbf{Z}_{t-2} \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\1 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{X}_{t-1} \\ \mathbf{Y}_{t-1} \\ \mathbf{Z}_{t-1} \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\1 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{X}_{t} \\ \mathbf{Y}_{t} \\ \mathbf{Z}_{t} \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\1 \end{pmatrix}$$

⑤ $$\begin{pmatrix} \\ \mathbf{X}_{t-2} \\ \\ \\ \mathbf{Y}_{t-2} \\ \\ \\ \mathbf{Z}_{t-2} \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \\ \mathbf{X}_{t-1} \\ \\ \\ \mathbf{Y}_{t-1} \\ \\ \\ \mathbf{Z}_{t-1} \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \\ \mathbf{X}_{t} \\ \\ \\ \mathbf{Y}_{t} \\ \\ \\ \mathbf{Z}_{t} \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Substituting these to equation (40), we obtain the following estimated Matrix.

$$\mathbf{P} = \begin{pmatrix}
2 & 3 & 2 & 5 & 8 & 6 & 0 & 1 & 0 & 1 & 2 & 1 & 5 & 4 & 1 & 0 & 0 & 0 \\
2 & 3 & 4 & 1 & 4 & 6 & 0 & 3 & 0 & 1 & 1 & 2 & 3 & 3 & 2 & 0 & 0 & 0 \\
4 & 4 & 5 & 3 & 7 & 4 & 0 & 0 & 0 & 2 & 2 & 3 & 5 & 5 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & 0 & 0 & 2 & 3 & 3 & 0 & 0 & 0 & 3 & 0 & 3 & 2 & 3 & 1 \\
0 & 0 & 0 & 3 & 5 & 0 & 5 & 2 & 5 & 0 & 0 & 0 & 3 & 1 & 2 & 3 & 3 & 3 \\
0 & 0 & 0 & 0 & 2 & 4 & 3 & 2 & 4 & 0 & 0 & 0 & 2 & 1 & 2 & 4 & 3 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & 4 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 5 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1
\end{pmatrix}$$

$$\times \begin{pmatrix}
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 17 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 26 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 10 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 20 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6 & 10 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 21 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 9 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 10 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 9 & 10 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 21 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 10 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 8 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13
\end{pmatrix}^{-1}$$

$$
= \begin{pmatrix}
0.250 & 0.200 & 0.200 & 0.522 & 0.751 & 0.912 & 0.016 & 0.087 & 0.040 & 0 & 0.200 & -0.033 & -0.430 & -0.561 & -0.802 & -0.036 & -0.070 & -0.045 \\
0.250 & 0.400 & 0.400 & 0.104 & 0.282 & 0.521 & 0.047 & 0.262 & 0.120 & 0 & -0.200 & -0.067 & -0.085 & -0.184 & -0.314 & -0.107 & -0.211 & -0.136 \\
0.500 & 0.400 & 0.400 & 0.130 & 0.174 & 0.130 & 0 & 0 & 0 & 0 & 0 & 0.100 & 0.087 & 0.155 & 0.029 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.151 & -0.304 & -0.497 & 0.062 & 0.086 & 0.132 & 0 & 0 & 0 & 0.270 & 0.376 & 0.715 & 0.093 & 0.112 & -0.042 \\
0 & 0 & 0 & 0.184 & 0.186 & -0.107 & 0.191 & -0.001 & 0.143 & 0 & 0 & 0 & -0.014 & -0.013 & 0.225 & 0.111 & 0.164 & 0.106 \\
0 & 0 & 0 & -0.007 & 0.125 & 0.277 & 0.009 & -0.063 & 0.066 & 0 & 0 & 0 & 0.012 & -0.119 & -0.085 & 0.337 & 0.231 & 0.112 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.289 & 0.340 & 0.231 & 0 & 0 & 0 & 0 & 0 & 0 & -0.130 & -0.301 & -0.021 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.120 & 0.275 & 0.126 & 0 & 0 & 0 & 0 & 0 & 0 & -0.076 & -0.079 & 0.082 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.111 & 0.228 & 0.215 & 0 & 0 & 0 & 0 & 0 & 0 & -0.149 & -0.009 & -0.132
\end{pmatrix}
$$

The Block Matrices make upper triangular matrix as is supposed. We can confirm that $\boldsymbol{K}_4, \boldsymbol{K}_7, \boldsymbol{K}_8, \boldsymbol{L}_4, \boldsymbol{L}_7, \boldsymbol{L}_8, \boldsymbol{M}_2, \boldsymbol{M}_3, \boldsymbol{M}_4, \boldsymbol{M}_6, \boldsymbol{M}_7, \boldsymbol{M}_8, \boldsymbol{N}_4, \boldsymbol{N}_7, \boldsymbol{N}_8, \boldsymbol{Q}_2, \boldsymbol{Q}_3, \boldsymbol{Q}_6, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{R}_4, \boldsymbol{R}_6, \boldsymbol{R}_7, \boldsymbol{R}_8$ are all 0. $\boldsymbol{M}_1, \boldsymbol{M}_5, \boldsymbol{M}_9, \boldsymbol{R}_1, \boldsymbol{R}_5, \boldsymbol{R}_9$ become diagonal Matrices as we have assumed.

6. Application of this Method

Consumers' behavior may converge by repeating forecast with above method and total sales of all brands may be reduced. Therefore, the analysis results suggest when and what to put new brand into the market which contribute the expansion of the market.

There may arise following case. Consumers and producers do not recognize brand position clearly. But analysis of consumers' behavior let them know their brand position in the market. In such a case, strategic marketing guidance to select brand would be introduced.

Setting in order the brand position of various goods and taking suitable marketing policy, enhancement of sales would be enabled. Setting higher ranked brand, consumption would be promoted.

7. Conclusion

Consumers often buy higher grade brand products as they are accustomed or bored to use current brand products they have. Focusing that consumers' are apt to buy superior brand when they are accustomed or bored to use current brand, new analysis method is introduced. Before buying data and after buying data is stated using liner model. When above stated events occur, transition matrix becomes upper triangular matrix.

In this paper, block matrix structure under brand groups was clarified when brand selection was executed toward higher grade brand. Equation using transition matrix stated by the Block Matrix was expanded to the third order lag and the method was newly re-built. In numerical example, matrix structure's hypothesis was verified. This approach makes it possible to identify brand position in the market and it can be utilized for building useful and effective marketing plan.

Such research as questionnaire investigation of consumers' activity in automobile purchasing should be executed in the near future to verify obtained results. Various cases should be examined hereafter.

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