The Number of Stocks Before (n) and After Portfolio Optimization (k): The Heuristic k ≈ sqrt(n)

Manuel Tarrazo[1]

[1] School of Management. University of San Francisco, San Francisco, California, USA

Correspondence: Manuel Tarrazo, School of Management. University of San Francisco, San Francisco, California, USA.

Received: February 11, 2024 Accepted: April 7, 2024 Online Published: April 9, 2024

doi:10.5430/afr.v13n2p32 URL: https://doi.org/10.5430/afr.v13n2p32

Abstract

This study focuses on the relationship between the number of securities (n) pre-selected for mean-variance portfolio optimization and the number of optimal securities (k). We propose a heuristic k ≈ square root (n) based on empirical research optimizing different sized (n) portfolios. That is, a sample selection of 20-30 securities should yield a portfolio of about five optimal securities, and an initial sample of 500 securities, should result in an optimal portfolio of about 22. We focus on the tangent portfolio that maximizes the return-to-risk ratio. The heuristic finds its support, rationale, and logic in the numerical properties and statistical nature of the optimization. More specifically, the heuristic seems to originate in the dynamic convergence patterns observable in many statistical processes, especially in standard deviations. It is also supported by available results in the literature. Our "square root" heuristic functions as part of the wider family of approximation around the power law, where some variables (authors, securities, people) receive a disproportionate share of a given collection of items – see, for example, Pareto's principle, Zipf's law, Lotka's law, Price's square root law, Simon's law, etc. The heuristic provides assistance not only in anticipating the number of optimal securities chosen by the mean-variance optimizer but also in suggesting selectivity in the effort of pre-selecting securities prior to the optimization and in sharpening portfolio-based approaches to investing in general. In sum, the heuristic k ≈ sqrt(n) seems helpful at all levels of portfolio management.

Keywords: heuristic optimization, portfolio optimization, mean-variance, Markowitz model, limited diversification

1. Introduction

"Empirical formulae are not laws of nature yet, but can be important steps towards the discovery of theoretically substantiated functional relationships between physical quantities. An almost classical example of it is the discovery of Planck's radiation law on the basis of three empirical formulae." (Bronshtein and Semendyayev, 1985, p. 729)

We studied the matter of the number of securities preselected before portfolio optimization (n) and the number obtained from the optimizer (k) when calculating optimal portfolio positions, i.e., optimal weights. Selecting which securities to buy is one of the two most critical parts of the application of portfolio optimization to actual investing; the other is making money. Despite its importance, and despite the generous amount of financial literature that addresses portfolio analysis, we have not seen any systematic study of the n-k relationship.

One possible reason for that void in the literature is the inherent difficulty in trying to comprehensively shed light on this issue. We demonstrate one effective way to proceed: follow the lead of the earliest formulations of portfolio optimization, and focus on a case of an individual, small investor implementing the two steps mentioned above.

Concerning portfolio construction, one enduring assumption is that the many resources available to large, institutional investors justify large numbers for n, the number of securities fed into the optimizer. Furthermore, numerical optimizations for large-scale problems are not as expensive/intractable as they used to be. Finally, institutional investors can analyze layers upon layers in the investing decision – the firms issuing the stock, the financial markets, the economy, the international scene, etc. To the contrary, we believe that excess resources and ingredients may be too many cooks spoiling the n-k broth.

Facing the same challenges, scholars have reacted similarly to large investors – throwing more resources at the problem and believing larger sizes and more complex tools would help. Specifically, scholars have concentrated their efforts in the following endeavors: 1) identifying areas from which to draw guidance (statistics, economics, financial markets, corporate financial analysis, etc.), 2) numerical optimization of a function based on certain parameters representing return and risk, and 3) evaluating the resulting portfolio. At every turn, scholars have come back to the problem with more resources and more impressive tools (mathematical programming, inequalities, auxiliary variables, theoretical probability, infinite variance, rocket mathematics), and impressive theories as well (mean-variance, Roy's safety-first and related ideas, levered-portfolios, CAPM, utility theory).

The barrier to progress in portfolio optimization is the lack of progress in understanding the n-k relationship. Being unable to justify that the optimal course of action is to invest in k optimal securities in exactly the calculated optimal weights prevents us from actually buying those securities.

It is telling the more hopeless the attempt has been, 1) the more it has clothed itself in the form of presumably irrefutable proofs, lemmas, theorems, and corollaries, and 2) the lower the acceptance to tackle the problem in to tackle the n-k matter with more straightforward ways. Any such attempt is shot down before take-off because, it is alleged, it does not promise to be "robust" or "rigorous" enough. And thus, the loop of portfolio investing is never closed.

We followed a direct path, and we have something to share with investors and our academic colleagues.

(1) We first performed both an extensive and intensive analysis of the literature, which in some ways is unbounded. First, many of the contributions regarding "portfolio size" referred to the size of funds invested, not the any number or any list of the assets involved, and their optimal weights. Second, much of the literature that addressed the issue of "how many stocks make a diversified portfolio" did not perform any optimization at all. A great many of them associated risk reduction with a theoretical minimum, CAPM-inferred market covariance level. Others, primarily work involving simulations, did not perform any optimizations. In the worst cases, a most illogical assumption (choosing securities at random), which no real investor would take seriously, led to similarly ill-conceived condition (equal position weights), and produced a final recommendation to – unsurprisingly – load up on securities by the hundreds.

(2) We then selected a specific portfolio optimization mode: the unlevered version of the original mean-variance model, as presented by Markowitz (1959).

(3) We kept the add-ons to a minimum: a) focus on one portfolio (the one maximizing the return-to-risk ratio; b) no risk-rate; risk analysis can be performed once the n-k relationship has been clarified; c) keep utility implicit, as in Constantinides and Malliaris (1995). We studied the minimum-variance portfolio as well.

(4) We worked at two levels: we focused on the small portfolio (ten securities) level to allow analysis of each optimization, but we also included several large portfolio groupings in our analysis.

(5) Fluctuations in the stock market affect the optimal number of securities in optimal portfolios. It can be shown that mean-variance portfolios optimizer acted like a thermostat, reducing the number of securities in optimal portfolios to keep the return-to-risk ration as high as possible, given the circumstances. We studied optimal portfolios (tangent and minimum-variance) during around the mid-2000s crisis.

Our work is organized in three sections. The first section is dedicated to the literature, the second to the material covered in points (2) to (5) above, and the third to our specific additions: numeric, quadratics, and the heuristic $k \approx \text{sqrt}(n)$.

In our analysis, we tried to detect or construct helpful rules-of-thumbs or heuristics concerning the n-k relationship. For example, if one inputs n securities into the optimizer, how many would it determine would be "optimal:" half of n, the square root of n? We determined $k = \text{sqrt}(n)$ is a useful heuristic, with interesting implications for investing practice.

Concluding comments and references close the study.

2. The Literature

The existing literature is quite extensive, which made it advantageous to develop a typology of what makes some studies are more helpful than others (e.g., both those with findings regarding n and k individually, and those with incidence on the n-k relationship), and also keep in mind those indirect areas that can clarify the n-k issue.

Ninety-nine percent of the literature pointed to Evan and Archer (1968) as the initial reference on the optimal number of stocks for a portfolio, which we designated k in our notation. This is important for two reasons. First, the earliest illustration of an application of mean-variance analysis was presented by Markowitz (1959, Chapter 2), and it is excellent; the model employed 9 stocks, assumed a risk-free rate paying cash, and covered 18 yearly return observations (1937-1954). At that time, the required optimization was hard to implement with the available computing power. Markowitz (1952) stated that he had intended to use 25 securities but gave up (Appendix: Personal Notes, pp. 381 and ss.). Second, the date of Evans and Archer's (1968) work indicates it likely influenced the development of the CAPM, and the authors write in the first paragraph, "The problems associated with portfolio analysis have been the subject of intense discussion in recent years, especially after the introduction of a practical solution algorithm by Sharpe…," (p. 761). Why would the authors turn to the CAPM when mean-variance analysis was still in its late teenage years? It was because the CAPM differentiated between systematic risk (unavoidable for any investment in stocks) and unsystematic risk, which was the real target of security measures added to the portfolio. Evans and Archer came to very positive conclusions, eight to 10 securities were all that was needed in the portfolio. They emphasized costs and the cost-benefits of adding securities, but their analysis included "randomly selected portfolios," where "equal dollar amounts were invested in each security in each of the portfolios, and dividends were not invested.

Kryzanowski and Singh (2010), however, point to an earlier indication of the optimal number of securities: Fama (1965), who used an example of Paretian market and suggested using 15 securities might work but at least 100 might be required to exhaust diversification benefits.

A study by Fisher and Lorie (1970) made multiple helpful contributions: they used wealth ratios, not returns, to avoid misleading return assumptions. They included commissions, introduced Gini and Lorenz curve statistics, and assumed equal investments without any subsequent reallocation of resources and concluded that 8 stocks would provide 80% of the maximum possible diversification benefit; 16, 90%; 32, 95%; and 128, 98%.

Mao (1970) indicated that a cost-benefit analysis was needed to provide an answer to the problem of the optimal number of securities needed, and he provided one using the expected return-to-beta ratio to cases where all securities had positive returns. This is very useful because it links the n-k relationship to the critical ratio at play in the optimization itself. We believe that using individual return-to-standard deviation ratio provided by mean-variance, rather than the one provided by the CAPM, would have been even better.

Jennings (1971) addressed a different important factor – the competence of professional portfolio managers and the industry to provide diversification: "One of the inherent limitations of a portfolio manager is his inability to evaluate an infinite number of securities. The seriousness of this problem is directly related to the risks associated with a 'small' portfolio. The economic function of a mutual fund industry is to provide diversification and professional management. If it is assured that a 'small' portfolio can virtually eliminate diversifiable risk, the necessity of these functions may be questioned. In addition, the strategy of concentration may be less 'risky' than is commonly supposed. Finally, the modern portfolio models generally assume that portfolio additions are costless" (p. 797). This is a very prescient observation, which sums up today's situation perhaps even better than it did when it was made. Today, small-portfolio investors have more funds, education, information, and computing power than those of 1971, and some invest in small portfolios they put together themselves. Jennings studied two strategies, buy-and-hold and yearly full replacement, with randomly selected stocks. His recommendation concerning the optimal number of securities was similar to Fama's (1965) above. Jennings also concluded that, "The characteristics of the investor will influence the number of securities contained in his portfolio. Diversification is not a 'free good.' The costs include brokerage fees, analytical and management expenses, and the opportunity for an exceptional portfolio gain that is assumed to decline as common stocks are added to the portfolio" (p. 799).

Wagner and Law (1971) highlighted the information hypothetical investors may use before buying the stocks and another key item: the information used to manage risk and its effects on the number of securities to buy. They built random portfolios of 1-20 securities and checked if stocks ratings from the S&P Guide affected both performance and the number of securities. They found a "rapid decline in total risk as a portfolio is expanded from one security up to ten securities… Increasing the number of holdings does not, in and of itself, increase or decrease the rate of return on the portfolio… Portfolios of a small number of securities are very undiversified, whereas portfolios of as few as fifteen to twenty securities have a strong relationship to the market index…. Investment performance can often be improved by expanding the list of qualified securities to include higher return, higher risk stocks, while offsetting the increase in market risk through more effective diversification. Small accounts should be encouraged to pool their assets to exploit these possibilities" (Our boldfacing, Wagner and Law, 1971, several pages).

Note, for further reference: "On March 4, 1957, the index was expanded to its current 500 companies and was renamed the S&P 500," (https://en.wikipedia.org/wiki/S%26P_500_Index#History). The square root of 500 is approximately 22.

Fielitz (1974) went back to using managed investments companies, "Should they be considered viable investment alternatives for individual investors? The answer is not obvious since these companies provide diversification and thereby reduce risk, which is an advantage of prime importance… Direct investment in a randomly diversified portfolio of common stocks is preferred since the return performance of the random stock portfolio is superior to that that can be obtained from investment funds on the average," (pp. 54 and 61). He recommends eight randomly selected stocks for individual stock portfolios, and 20 selected at random to approximate the market portfolio (again, a number close to the square root of 500). He also notes, "[T]he time is near when 'index' funds will be available to the general public."

Jacob's (1974) study is pivotal in the literature on household investing. Realizing that small investors would not be able to buy many stocks, she developed a "limited diversification" approach, with which investors could find success from investing in as few as six securities. Brennan (1975) followed up by creating a single-index model in which the number of securities a variable that was dependent on costs. In his view, costs were the variable likely to keep small investors in the limited diversification area. Blume and Friend (1975) furthered this research by analyzing the characteristics of individual investors. Finally, Elton and Gruber made many contributions that small investors could make use of (e.g., simplified optimization procedures, clarifications on what matters when studying diversification, etc.); their (1977) research provides an analytical method to study the relationship between securities and portfolio risk.

Goldman's (1979) contribution may be considered very indirect, as he did not specify any particular number of securities and was responding to highly theoretical issues concerning Merton-type assets and stochastic calculus. But he highlighted the very important contrast between the concepts of "favoritism" and "diversification." The title of his study, "Anti-Diversification or Optimal Programs for Infrequently Revised Portfolios," anticipated very sophisticated modern topics such as rational inattention and indicated why some investors might stick to small portfolios.

Sengupta and Sfeir (1985) addressed over and under diversification by focusing on statistical efficiency – weights reliability. The matrices used in the optimization were negatively affected differences between the number of observations (rows) and the number of securities (columns). Increasing the number of observations for a given number of variables makes the optimal weights less reliable through overdetermination; increasing the number of variables causes underdetermination. Restricting both makes for more robust estimates and protects from nonnormality concerns. In addition, the variance of returns may not adequately measure the risk of a portfolio. Bird and Tippett (1986) followed up on the previous study and assessed portfolios by varying the number of observations and the number of variables. In this context, so-called "naive diversification" – piling up equally weighted, randomly selected securities – may be most detrimental, even for portfolios with as few as eight securities. They cautiously suggested, at the very least, doubling the number of stocks recommended by Evans and Archer (1868) to 16-20 instead of 8-10. More recently, Goetzmann and Edwards (1994) studied the issues using short-horizon data to build portfolios that could be held for long horizons, which touches upon the issues raised by Sengupta and Sfeir (1985) discussed above. DeMiguel et al. (2008) tried to measure the inefficiency of "naïve diversification."

Statman (1987) used a levered-portfolio context to set the number of stocks to be held at 30 when borrowing and 40 when lending. Woerheide (1993) provided useful diversification indices, and Newbould and Poon (1993) examined textbooks and found the recommendations varied between 8 and 20 securities. Beck et al. (1996) found that "substantial" diversification benefits were realized when the portfolio size approached 20 securities – note that, as in most studies, securities were chosen at random. DeWitt (1998) addressed with lack of information and made the case for naïve diversification – adding securities at random and using equal weights. He concluded by conjecturing that "100 to 500 securities might still lead to economically significant reductions in the required excess return," leaving the reader wondering how or why. Xu and Malkiel (2003) recommended 50 stocks on the basis of idiosyncratic volatility.

In the 2000s, a climate developed in which individual investors could be more successful than previously assumed. Vissing-Jorgensen (2003) found that many behavioral finance objections disappeared when individuals had enough wealth to allow appropriate investment (see Goetzmann and Kumar (2008) for an example of behavioral finance application). Chhabra (2004) wrote, "[T]here is a remarkable group of people, a minority of the general population but a significant majority of the affluent, who have become wealthy over the years by defying market diversification. Ironically, many of them would not have become so wealthy had they followed a conventional diversified approach" (p. 8). Riepe (2004) noted the lack of specifics in many studies advocating for diversification, which he likened to an "eat your vegetables mantra." However, in other studies, the recommended number of stocks continued to increase: one suggested 30 securities randomly selected from industry leaders might achieve diversification equivalent to that of the market index (Boscaljon et al., 2005); another noted more than 100 securities might be needed when shortfall risk was considered (Domian et al., 2007); and another recommended 40-50 stocks (Bellenjoun, 2010).

Two studies provide a close to our review of the literature. Part of Ivkovick et al.'s (2008) abstract aptly summarize their findings: "Stock investments made by households that choose to concentrate their brokerage accounts in a few stocks outperform those made by households with more diversified accounts (especially among those with large portfolios). Excess returns of concentrated relative to diversified portfolios are stronger for stocks not included in the S&P 500 index and local stocks, potentially reflecting concentrated investors' successful exploitation of information asymmetries. Controlling for households' average investment abilities, their trades and holdings perform better when their portfolios include fewer stocks" (p. 613).

Kryzanowski and Singh (2010) provided an excellent integration of the previous literature and carried out additional research using simulations (5,000 portfolios, Monte Carlo approach, equal weights). One major finding was, "mixed evidence for the refutation by various authors … that investors suffer from the cognitive error that losses are reduced over longer holding periods. Specifically, while the probability of not earning a positive return is reduced with a longer holding period, the probability of not earning the market return is increased with a longer holding period. Thus, the existence of cognitive error depends upon the choice of anchor or benchmark return" (p. 29). Another finding was that "the minimum portfolio size depends upon the chosen investment opportunity set, the metric(s) used to measure the benefits of diversification, and the criterion chosen to determine when the portfolio is sufficiently well diversified" (op. cit., p. 1).

More recently, the very comprehensive bibliometric study by Zaimovic et alia (1921, p. 551) note that, "a generalized optimal number of stocks that constitute a well-diversified portfolio does not exist for whichever market, period or investor."

What do we learn from the literature review? Negative observations include the following:

(1) Very little has been established about how to pre-select securities ahead of optimization (that is, how to set n). A few authors suggested focusing on an index might be useful.

(2) Authors rarely used optimal portfolios to set k, but they relied on observing given metrics for risk when the number of securities held for long periods changed, and these grouping almost invariably consisted of randomly grouped stocks. In other words, one cannot find empirical studies reporting findings along these lines: "We selected these n stocks, ran the optimization, obtained k optimal stocks, and this is what the follow up shows for returns and risks."

(3) In addition, without mean-variance optimization, one cannot evaluate the only quantifiable variety of diversification we know about, which is related to splitting the portfolio variance into individual security variances and covariances. Chasing after the conjecture market-covariance minimum is not comparable because of Roll's critique – the double implication of both mean-variance analysis and market theory hold true. Diminishing investing risk by splitting funds between a portfolio of risky securities and cash is hedging not diversification.

(4) Finally, there are some very important issues not included in a buy and hold, single-period, static problem. Some authors referred to the evolution of the investors' knowledge and financial position, which may affect to the length of the investing period and window of opportunity during which the investment takes place. Martellini and Urosevic (2006) found that optimal solutions produced by mean-variance analysis might not be as optimal under uncertain investment exit times.

On the positive side, our examination of the literature stresses two most useful points:

(1) Portfolio optimization may benefit from studying the portfolio econometric properties. Prior to Sengupta and Sfeir's (1985) study, Jobson and Korkie (1983) showed how to optimize a mean-variance portfolio using an atypical regression, where the matrix of returns was regressed on a column of one's (any constant would do), without using an intercept. Rebalancing the beat estimates produced the optimal weights, as well as standard errors for the regression and each of the coefficient. The procedures also suggested a very handy algebraic model that simplified both the optimization itself and its analysis (see Tarrazo, 2009, 2014, 2018).

(2) Kryzanowski and Singh (2010) noted, "the minimum portfolio size depends upon the chosen investment opportunity set, the metric(s) used to measure the benefits of diversification, and the criterion chosen to determine when the portfolio is sufficiently well diversified," (p. 1). This means that if we provide a logical way to prelect "n," the securities to be optimized, mean-variance analysis will provide the additional elements needed to study the n-k relationship. As previously done in the literature, we used several indices as the initial samples for the optimizations.

Markowitz's (1952, 1959) mean-variance model appears ideal to study the n-k relationship. It makes sense of a) investing by concentrating efforts in two dimensions: risk and return; b) it also makes sense at the statistical level – it employs well-known, easy-to-calculate descriptive stats (mean and variance) within the context of the normal distribution. And it shows that the optimal portfolio is also an optimal predictor. The numerical aspects that seemed prohibitive when it was proposed have become very reasonable and need no involve mathematical programming at all (see Tarrazo, 2014).

Other items like sample size and investment windows (optimal horizons) still remain on the researchers' back burners. But this is where our study comes in. Our analysis and contributions will be as follows:

(1) In the next section, we first study the effect of markets and the number of securities in the three critical portfolios in the efficient frontier for different portfolio groupings: the minimum-variance portfolio (portfolio with minimized standard deviation), the tangent portfolio (portfolio with the highest return-to-standard deviation), and the portfolio with the maximum return (portfolio with 100% of investment allocated to the security with the highest return).

(2) We present a different analysis using small portfolios -- ten pre-optimization stocks.

(3) Results from the previous point also clarify some matters related to sample length and horizon issues.

(4) We elucidate the major point of risk reduction, the number of securities, and the split between individual variance and covariance risk. We found it is more advantageous to diminish portfolio risk using financial analysis to pre-select securities than using blind security accumulation.

In addition to our empirical analysis of pre- and post-optimizations effects on the number of securities, we also, for the first time, explore the effects that the tools employed in mean-variance optimization have themselves on the number of post-optimal securities. This section provides an interesting and apparently valid and helpful heuristic to approximate the optimal number securities for the minimum-variance and tangent portfolios, given a pre-optimization number of n securities.

3. Empirical Analysis

The broad lines of mean-variance portfolio analysis, as in Markowitz's (1952, 1959) optimization are well known. The following paragraphs rely on work by Fabozzi, Gupta, and Markowitz (2002) and Constantinides and Malliaris (1995).

Investments were taken to be probabilistic variables, and their statistical measures or central tendency (average) and dispersion (variance, covariances) approximated the investments return (mean) and risk, respectively. These statistics were calculated by, first, collecting closing prices (adjusted for dividends and splits, five years' worth of monthly data) and, second, by computing the indicators using returns (natural logs of price relatives). The, the investor tried to maximize expected portfolio returns for a given level of risk or minimize risk for a given level of return. The optimization program can be stated in a number of ways – in terms of mathematical (quadratic) programming, in terms of Lagrangian optimizations, or in an algebraic manner, which we used in the calculations that follow.

Let,

$$F(w, V, r) = -1/2\ w'V\ w + w'r \tag{1}$$

represent the portfolio optimization problem, where w is a (k by 1) vector of optimal portfolio weights, V is the variance-covariance (k by k) matrix, and r is a (k by 1) vector of returns. Instead of using (1), we employed an auxiliary formulation, Z(z, V, r), where z (k by 1) is a vector of auxiliary variables. To solve Z, we compute z = inv(V) r , then we computed optimal portfolio weights by normalization, wi = zi / sum(z), and we used the optimal weights to compute portfolio returns, rp = w' r and portfolio variance, pvar = w'V w. The algebraic expression in (1) implicitly incorporates the investor's (quadratic) utility (Constantinides and Malliaris, 1995), which eliminates the need to handle another mystifying curve.

The work of Jobson and Korkie (1983) established that optimal mean-variance portfolios can be calculated using regression methods (see also Tarrazo, 2014, 2009; Britten-Jones, 1999; and Rekenthaler, 1999). MATLAB's function lsqnonneg(), provided a handy way to calculate optimal, non-negative ($w_i \geq 0$) portfolio weights, especially for large-scale portfolios. The most straightforward way to perform these calculations is to use the returns matrix and a column of ones for the tangent portfolio and the covariance matrix and a column of ones for the minimum variance portfolio. We used EXCEL's optimizer for the analysis of the small portfolios presented in the second part of this section.

We were interested in the optimal number of securities for the three key portfolio in the efficient frontier: 1) maximum returns (100% into the security with maximum returns), 2) the tangent portfolio (that maximizing portfolio return to portfolio standard deviation, rp/pstd), and 3) the minimum variance portfolio.

We first present our analysis of the n-k relationship in large portfolios under changing market conditions.

3.1 Large Portfolios and Market Effects

Anyone who has run portfolio optimizations for a few years will have noticed that the difference between the number of securities pre-selected for mean-variance portfolio optimization (n) and those within the optimal portfolio (k) seems to change – it gets narrower in good market times and significantly wider during poor market times. This effect can be observed simply by optimizing the Dow-Jones for different periods. Tarrazo (2018) carried out a more thorough analysis and evaluated the behavior of mean variance during the period 01/31/2001 to 12/30/2011, which included 132 monthly data points. The first optimal portfolio was calculated using 60 observations, five years of monthly data, and was dated 2006-01-3. Rolling portfolios were calculated every month after that first one for a total of 72 portfolios (non-negative weights). The data collected were grouped into the following four sets, corresponding to 1) companies in the Dow Jones Industrial Average (DJIA; 30 stocks), 2) companies in the NASDAQ 100 index (ndx100; 81 stocks), 3) stocks of companies in the Standard & Poor's 100 Index (sp100; 93 stocks with options traded), and 4) a set of stocks of companies from the San Francisco Bay Area (baf; 131 stocks), mostly tech-intensive businesses (note: R = rows = observations; and C = columns = variables).

(1) djia-092112-4matlab 132R by 30C

(2) ndx100-1012-4matlab-alpha 132R by 81C

(3) sp100-1012-4matlab 132R by 93C

(4) baf-1012-h-alpha-4matlab 132R by 131C

This methodology let us observe how mean-variance optimization worked and performed during this extended time period, which included the late 2000s economic/stock market crisis.

According to the National Bureau of Economic Research (NBER), "A recession is a significant decline in economic activity spread across the economy, lasting more than a few months, normally visible in real GDP, real income, employment, industrial production, and wholesale-retail sales" (https://en.wikipedia.org/wiki/Recession). NBER provided specific dates for the beginning and end of the crisis: December 2007 – June 2009, 18 months (Portfolios 24 and 42, respectively). (https://admin.nber.org/cycles/main.html). On September 29, 2008, the Dow Jones Industrial Average fell 777.68 points. On March 6, 2009, the Dow dropped to its lowest, closing at 6,594.44, a total decline of 53.4% from its peak close of 14,164.43 on October 9, 2007.

Figure 1 includes charts showing DJIA returns and portfolio standard deviations for the tangent portfolio during the period of reference. These polar charts that work like walking down a spiral staircase, from the highest point (Portfolio 1, January 3, 2016) to the lowest (Portfolio 72, December 30, 2011). The solid line depict portfolio returns and the broken line indicates portfolio standard deviations; portfolio returns resemble a crown close to the north pole-like center, for some of the groupings (DJIA, BAF) more than for others. The portfolio standard deviations seem to spiral out with frightening centrifugal force (see Tarrazo, 2018, for the analyses of the other three groups). The return-to-risk ratio also exhibits deterioration, as is demonstrated in the chart showing the ratio for the DJIA (bottom of the exhibit) and is covered in more detail in the next exhibit. It is hard not to conclude that mean-variance optimization had to do with preserving the levels of return much like a thermostat preserves the temperature in a house.

Figure 2 illustrates how portfolio optimizations countered deteriorating standard deviations and risk-to-return ratios to preserve and stabilize levels of returns – by reducing the number of optimal securities (k). Each row shows a different portfolio DJIA (30), S&P100 (93), NDX (81), and BAF (131). The numbers within the parentheses, the actual number of securities in the sample, were the values for n, the pre-optimization number of securities in each of the portfolios. In the chart on the left, each row shows the return-to-standard deviation ratio (pr/pstd) for each of the critical portfolios in the efficient frontier during each rolling month optimization.

Several items are noteworthy:

(1) The 2007 recession is clearly observable in all charts. It is even more remarkable that the charts' peaks and recovery coincide so well with the NBER dates for the beginning and end of the recession, Portfolios 24 and 42, respectively.

(2) The tangent portfolio in each grouping reaches higher return-to-risk ratios (and these are ex-post as well) than the minimum-variance portfolio, and the ratio of the security to the largest return meanders curiously between them.

(3) The difference in the ratios remains stable throughout the period for each grouping, which is also quite remarkable and signals the effects of the optimization.

(4) The right column shows the minimum-variance portfolio holding more stocks than the tangent portfolio for each of the groupings.

(5) The difference in the number of securities in each optimal portfolio is not as stable as that of the ratios.

(6) Each chart clearly depicts three regimes: pre-crisis, crisis, and post crisis (more details about this later). Each regime exhibits some stable (flat) range for the number of securities in each optimal portfolio, and further analyses of these trends are presented in a subsequent section.

In sum, there seems to be a relationship between the pre- and post-optimal number of securities in optimal portfolios. The evidence present so far suggests that this relationship is determined by economic/market conditions. In a following section, we show that it is also determined (in general) by the mathematical nature of the tools employed.

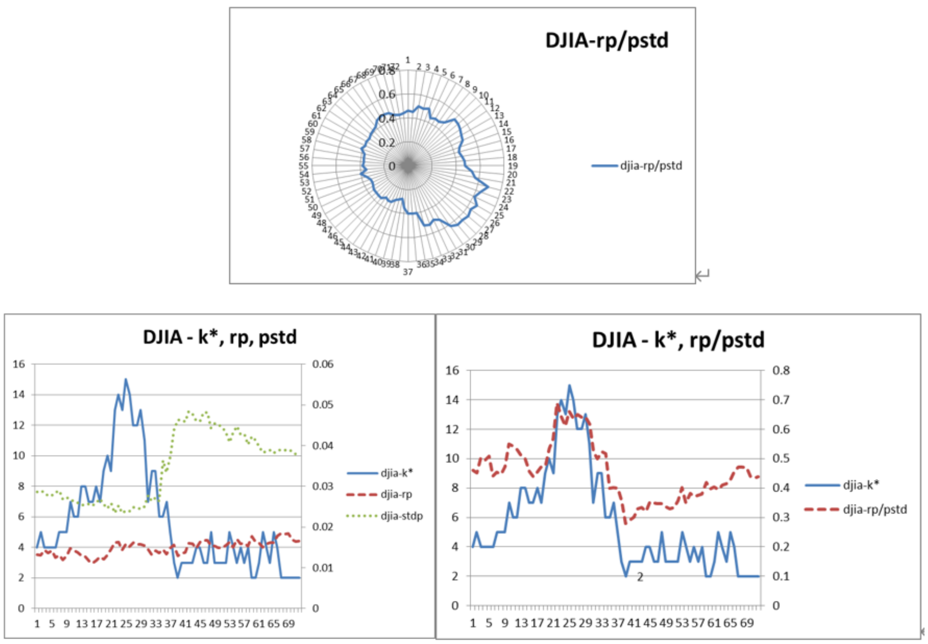


Figure 1. DJIA optimal portfolios: January 3, 2016 -- December 30, 2011

Source: Tarrazo, M. “Mean-Variance under Stress -Ten years of Portfolio Optimizations: 2001-2011.” Quarterly Journal of Finance and Accounting, Vol. 50, Issues 1 & 2, Winter & Spring 2018, 91-148.

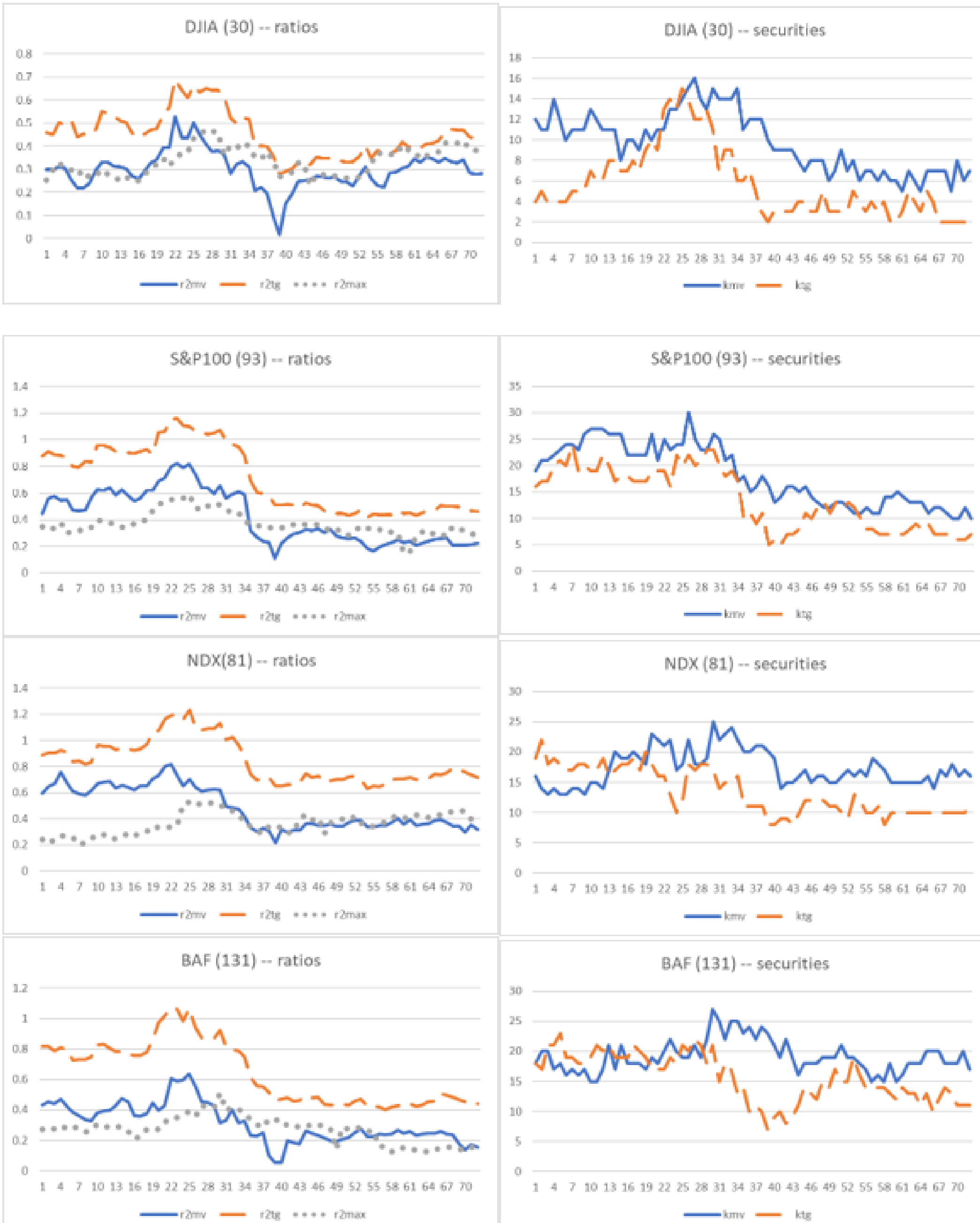


Figure 2. Efficient frontiers, return-to-risk ratios and number of securities

Source: Tarrazo, M. "Mean-Variance under Stress -Ten years of Portfolio Optimizations: 2001-2011."

Quarterly Journal of Finance and Accounting, Vol. 50, Issues 1 & 2, Winter & Spring 2018, 91-148.

3.2 Small Portfolios

Figures 3 and 4 present optimizations (tangent portfolio, max rp/pstd, non-negative weights) for fourteen small portfolios: 10 securities pre-optimization, with five-years of monthly returns, for 03/01/2014-02/01/2019. These portfolios were put together as part of the normal analysis of the company issuing the stock (usual, textbook financial analysis concerning debt, ROE and ROA rates, etc.) and the stock price itself. All companies were attractive in several different ways.

The following observations offer insights on the n-k relationship:

(1) Note the boxes enclosing the securities that were admitted into the optimal portfolio. The number of optimal securities in each portfolio hovered around 5, that is n = 10 and k = 5. However, a deeper look reveals a significant amount of variation in optimal weights, with a pattern of the top three companies in each portfolio getting from 70-80% of the total portfolio weight (100% in two cases) and the top four, from the mid-80 to 100% in some cases, as reflected in the columns to the right of the optimal weights. In other words, all the action centered around three to four securities.

(2) The individual security ranking (individual security return to individual security standard deviation ri/pstdi) and the optimal weights were directly related. The ranking of the securities in the optimal portfolio (those in the box) followed nearly exactly the individual return-to-risk ratio. That means the optimizations rankings reflect individual security rankings very closely. In the cases represented by the straight arrows, the ranking was identical. In the cases indicates by the inclined arrows, the ranking was approximate. The securities with lower individual rankings have little hope of getting themselves admitted to the "the club" represented by the optimal portfolio.

(3) The arrows offer a qualitative representation of diversification: positively correlated securities that were demoted are represented with downward pointing arrows; and negatively correlated securities that were promoted have upward pointing arrows. Quantitatively, the effect can be ascertained by looking at the weight gain (loss) in each case. For example, in Portfolio 1, the individual return-to-risk ranking imposed itself almost completely, with the exception of the 5.55% Apple lost to Facebook. Individual return-to-risk coefficients determined membership in the optimal portfolio, and diversification (covariance) effects re-arranged where those securities would sit at the tables.

(4) Based on the rp/pstd ratios reported at the top of each portfolio, ten securities seemed to be enough to earn an impressive rate of risk-adjusted returns.

Some readers may be surprised to read those individual effects (main diagonal, direct effects, variance) prevailed over indirect (off-diagonal, covariance) effects. In other words, that mean-variance optimization was predominantly about individual return-to-risk proportions. But this is a mathematical characteristic of the model itself. It happens because the variance-covariance matrix is positive and definite. As Mao (1970) showed, for diagonal matrices (no covariance terms), the ranking of optimal weights follows the ranking of individual return-to-risk ratios exactly: that is, wi = (ri/vari) / (Σ ri/vari), where vari is the individual security variance. This is readily apparent in the diagonal variance-covariance matrix of Mao's (1970) case, but it is harder to see in a positive definite matrix. As Golub and Van Loan (1996) put it, "A symmetric positive definite matrix has a 'weighty' diagonal. The mass on the diagonal is not as blatantly obvious as in the case of diagonal dominance but it has the same effect in that it precludes the need for pivoting," Golub and Van Loan (p. 141). In our case, this means the main diagonal imposed its ranking over off-diagonal terms, in conjunction with the corresponding rate of return.

Because of the ri/stdi ranking, no security with negative returns would be ever admitted in a portfolio that has positive returns. Because of linearity, the return-to-risk ratio of the portfolio equaled the ratio of individual securities, marginal returns to individual securities marginal variance. The denominators are all nonnegative, so the numerator of the individual, marginal return must be positive if the portfolio return is positive. Though a necessary condition, having a positive return alone is not sufficient for a security's inclusion in the optimal portfolio. The security also has to improve the overall return-to-risk of the portfolio, which was very tough.

Our observations concerning diversification sharply conflicted with its portrayal in some of the literature as a powerful force that can be harnessed by indiscriminate accumulation of securities. We observed optimizations in which the optimizer ruthlessly discarded dozens of securities. In a well-selected portfolio, diversification might make large-return gains more stable. Diversification effects depend on the specific grouping into which each security is cast.

These results are most interesting and have the clearest relevance for practical investing. They indicate that mean-variance optimization, when properly applied, may uncover forces at play in the investment world. It may, in fact, tend to reduce the number of securities needed in an optimal portfolio.

At this point, we have seen there is an observable relationship between the pre-optimization number of securities and that of securities included in the optimal portfolio (tangent, and minimum-variance). In our next and final section, we present general relationship derived from the mathematical objects used in the analysis.

P1 rp/stdp 0.448666

	ri/stdi		wi*	
ADBE	0.395711	ADBE	38.69%	38.69%
INTU	0.357916	INTU	33.94%	72.64%
NVDA	0.322061	NVDA	13.86%	86.49%
INTC	0.252938	INTC	7.96%	94.45%
APPL	0.233182	FB	5.55%	
CRM	0.229921	CRM	0	
FB	0.215065	SAP	0	
GOOG	0.184156	APPL	0	
SAP	0.120186	GOOG	0	
CVX	0.068401	CVX	0	

P2 rp/stdp 0.345639

	ri/stdi		wi*	
CSCO	0.294263	CSCO	38.88%	38.88%
INTC	0.252938	INTC	31.53%	70.41%
CRM	0.229921	GOOG	14.98%	85.39%
GOOG	0.184156	CRM	14.61%	100.00%
SCHW	0.131283	CVX	0	
HPQ	0.130251	GPS	0	
WSM	0.130251	BMRN	0	
CVX	0.068401	HPQ	0	
BMRN	0.022418	WSM	0	
GPS	-0.0628	SCHW	0	

P3 rp/stdp 0.337578

	ri/stdi		wi*	
INTC	0.252938	INTC	46.61%	46.61%
CRM	0.229921	FB	29.13%	75.74%
FB	0.215065	CRM	24.26%	100.00%
GOOG	0.184156	CVX	0	
ORCL	0.120788	GOOG	0	
CVX	0.068401	TWTR	0	
WFC	0.065652	ORCL	0	
GPS	-0.0628	GPS	0	
TWTR	-0.0679	WFC	0	
YELP	-0.1022	YELP	0	

P4 rp/stdp 0.417866

	ri/stdi		wi*	
NVDA	0.322061	CSCO	49.06%	49.06%
CSCO	0.294263	NVDA	26.71%	75.76%
NFLX	0.228906	GHDX	17.42%	93.18%
GHDX	0.164847	NFLX	6.82%	100.00%
SSD	0.129527	EXEL	0	
EBAY	0.090091	GILD	0	
EXEL	0.082611	BEN	0	
WFC	0.065652	WFC	0	
GILD	-0.0314	EBAY	0	
BEN	-0.10703	SSD	0	

P5 rp/stdp 0.575406

	ri/stdi		wi*	
WCN	0.482923	WCN	51.05%	51.05%
CSCO	0.294263	CWT	21.20%	72.25%
CWT	0.259956	CSCO	14.09%	86.35%
AAPL	0.233182	NFLX	9.44%	95.78%
CRM	0.229921	AAPL	3.54%	
NFLX	0.228906	TSLA	0.68%	
EBAY	0.090091	YELP	0	
CVX	0.068401	CVX	0	
TSLA	0.041844	CRM	0	
YELP	-0.1022	EBAY	0	

P6 rp/stdp 0.414188

	ri/stdi		wi*	
NVDA	0.322061	LMT	27.38%	27.38%
CSCO	0.294263	CSCO	21.94%	49.32%
LMT	0.262436	INTC	21.56%	70.89%
INTC	0.252938	NVDA	14.94%	85.82%
AAPL	0.233182	ABM	7.28%	
CRM	0.229921	TSLA	5.10%	
ORCL	0.120788	AAPL	1.80%	
ABM	0.103957	ORCL	0	
CVX	0.068401	CRM	0	
TSLA	0.041844	CVX	0	

P7 rp/stdp 0.481861

	ri/stdi		wi*	
ADBE	0.395711	ADBE	36.18%	46.61%
V	0.373323	ROST	24.20%	75.74%
INTU	0.357916	INTU	23.20%	75.74%
ROST	0.278786	V	8.13%	
AAPL	0.233182	ISRG	4.89%	
CRM	0.229921	NFLX	2.05%	
NFLX	0.228906	EA	1.35%	
EA	0.228506	CRM	0	
AMD	0.180101	AMD	0	
ISRG	0.100269	AAPL	0	

P8 rp/stdp 0.506429

	ri/stdi		wi*	
ADBE	0.395711	ADBE	44.58%	44.58%
DLR	0.27356	CLX	26.57%	71.15%
CLX	0.241593	DLR	23.19%	94.33%
GHDX	0.164847	ISRG	2.69%	97.03%
ATVI	0.156425	BLUE	1.52%	
IRBT	0.142902	GHDX	1.46%	
RHI	0.139849	RHI	0	
BLUE	0.138151	IRBT	0	
ISRG	0.100269	ATVI	0	
CVX	0.068401	CVX	0	

Figure 3. Small portfolios I (cases 1-8)

Source: Tarrazo, M. "Mean-Variance under Stress -Ten years of Portfolio Optimizations: 2001-2011." Quarterly Journal of Finance and Accounting, Vol. 50, Issues 1 & 2, Winter & Spring 2018, 91-148.

P9 rp/stdp 0.417017

	ri/stdi		wi*	
NVDA	0.322061	CSCO	32.62%	32.62%
CSCO	0.294263	NVDA	21.67%	54.30%
INTC	0.252938	INTC	20.63%	74.93%
AAPL	0.233182	FB	20.16%	95.09%
CRM	0.229921	NFLX	4.91%	
NFLX	0.228906	AAPL	0	
FB	0.215065	GOOG	0	
GOOG	0.184156	WFC	0	
CVX	0.068401	CVX	0	
WFC	0.065652	CRM	0	

P10 rp/stdp 0.428288

	ri/stdi		wi*	
V	0.373323	V	68.12%	68.12%
NVDA	0.322061	NVDA	21.53%	89.64%
AAPL	0.233182	FB	8.18%	97.82%
CRM	0.229921	NFLX	2.03%	99.86%
NFLX	0.228906	AAPL	0.14%	
FB	0.215065	CVX	0	
GOOG	0.184156	GOOG	0	
SCHW	0.131283	CRM	0	
CVX	0.068401	SCHW	0	
WFC	0.065652	WFC	0	

P11 rp/stdp 0.514545

	ri/stdi		wi*	
ADBE	0.395711	CLX	31.04%	31.04%
NVDA	0.322061	ADBE	28.49%	59.54%
CSCO	0.294263	ESS	14.51%	74.05%
CLX	0.241593	NVDA	11.32%	85.37%
ESS	0.239473	COO	7.51%	
CRM	0.229921	CSCO	7.12%	
COO	0.220686	CVX	0.00%	
BIO	0.18928	COHR	0.00%	
COHR	0.097755	BIO	0.00%	
CVX	0.068401	CRM	0.00%	

P12 rp/stdp 0.293102

	ri/stdi		wi*	
LMT	0.262436	LMT	34.74%	34.74%
INTC	0.252938	INTC	27.26%	62.00%
AAPL	0.233182	WMT	12.00%	74.00%
EA	0.228506	EA	10.62%	84.62%
GOOG	0.184156	AAPL	9.57%	
WMT	0.128383	TSLA	5.81%	
ORCL	0.120788	GOOG	0	
MET	0.045104	ORCL	0	
TSLA	0.041844	MET	0	
AMC	-0.03333	AMC	0	

P13 rp/stdp 0.49994

	ri/stdi		wi*	
INTU	0.357916	CWT	30.77%	30.77%
NVDA	0.322061	CSCO	27.13%	57.89%
CSCO	0.294263	INTU	18.90%	76.79%
CWT	0.259956	NVDA	14.74%	91.53%
BIO	0.18928	GDDY	8.47%	
MGCR	0.164649	EXEL	0	
GDDY	0.113155	MGCR	0	
FORM	0.100937	BIO	0	
EXEL	0.082611	FORM	0	
EHTH	0.008268	EHTH	0	

P14 rp/stdp 0.440994

	ri/stdi		wi*	
ADBE	0.395711	ADBE	39.87%	39.87%
NVDA	0.322061	CSCO	20.69%	60.56%
CSCO	0.294263	NVDA	14.30%	74.87%
INTC	0.252938	FB	9.58%	84.45%
ALIGN	0.236966	INTC	8.24%	
AAPL	0.233182	JPM	7.32%	
JPM	0.222587	ALIGN	0	
FB	0.215065	AAPL	0	
GOOGL	0.181667	GOOGL	0	
AMAT	0.149885	AMAT	0	

Figure 4. Small portfolios II (cases 9 to 14)

Source: Tarrazo, M. "Mean-Variance under Stress -Ten years of Portfolio Optimizations: 2001-2011."

Quarterly Journal of Finance and Accounting, Vol. 50, Issues 1 & 2, Winter & Spring 2018, 91-148.

4. Numerics, Quadratics, and the Heuristic k ≈ sqrt(n)

This section explains a) the search for and development of the heuristic, b) the conceptual integration of the relationships between portfolio return-to-risk ratios for critical portfolios along the efficient frontier and the number of securities in each of those portfolios, c) the numerical integration of those two relationships, and d) an empirical assessment of the usefulness of the heuristic we developed.

The presence of patterns in the number of optimal securities in critical portfolios – the minimum-variance and the tangent portfolio – that are obtained from any given n, suggested the existence of an heuristic, which has been described as a rule of thumb, a quick and frugal approximation of practical value, or a short cut.

In some areas of research, heuristics have bad connotations (e.g., behavioral finance). In most of those negative cases, shortcuts are used to remedy something being done wrong due to 1) not gathering enough information (due process), 2) discarding available useful information, 3) not properly quantifying, or at least weighing, information, and 4) making decisions too quickly. Heuristics are only useful if they help in decision making. In our opinion, decision-making shortcuts originating from wrongful omissions and commissions should simply be referred to as errors or biases not heuristics. Useful, which should also mean proper, heuristics imply straight-forwardness, often built on the most important factors in the problem being solved. However, more attention is often paid to those errors or biases, as indicated by Wikipedia's "List of cognitive biases," which include cognitive, memory, and social biases and total almost one hundred entries, referring to many others in the "See also" section. The literature on the errors seems to grow exponentially all across disciplines (see Fonseca et al, 2002). But also note that each psychological habit may have a flipped side that is beneficial and] some behavioral biases appear to conflict with one another. A few guidelines, including "[to] develop quantitative investment criteria," seem useful in overcoming psychological biases in some important cases, see Baker and Nofsinger (2002, p. 112 and ss.).

The value of shortcuts is determined by where they originate, the problem they address, and they ways they are used by the decision-maker. Relative to systemic portfolio optimization that allows the implementation of a real investment plan, heuristics are useful because each part of the decision-making process – defining the problem, assessing the situation and goals, and understanding the investing opportunities (e.g., a model)—are overwhelming.

In our area of research (decision sciences, operational research), heuristics are not viewed second-class solutions. They have been be formally described as "A method which, on the basis of experience or judgement seems likely to yield a reasonable solution to the problem, but which cannot be guaranteed to produce the mathematical optimal solution" (Silver, 2004, p. 936). Heuristics may be qualitative or quantitative, and they are sought after to strengthen and improve the efficiency and accuracy of mathematical programming methods (see Ball, 2011). In fact, as noted by Geoffrion (1976), heuristics reflect the fact that effort is put into modeling and optimizing is to gain insight rather than to produce numbers. Simon and Newel (1958), writing at the peak of what Paul Samuelson has named one of the golden ages of economics (theory of inequalities, linear and nonlinear mathematical programming, game theory, mean-variance portfolio analysis), referred to heuristics as "the next advance in Operations Research." In addition, heuristics can bridge qualitative and quantitative information, a benefit receiving significant current interest (see the survey by Smith and Winterfeldt, 2004, and the effort in mapping attributes and choices in Hogarth and Karelia, 2005). Interestingly, Astebro and Elhedhli's (2006) study on assessing commercial success in early-stage ventures used a heuristics-based sorting procedure methodology roughly equivalent to a portfolio selection model.

Our review of the portfolio optimization literature showed us that research has either implicitly or explicitly employed heuristic reasoning and techniques, for example, Evans and Archer's (1968) suggestion to invest in eight to 10 securities. Jacob (1970) used CAPM betas as short cut in implementing "limited diversification" portfolios for individual, small investors. While looking for the optimal number of securities to hold, Mao (1970) searched for the essential variable or ratio that could determine diversification. In the same vein, Elton, Gruber, and Padberg developed simple rules and techniques for portfolio optimization, including assuming that the pairwise correlation coefficients between stocks was a constant number (e.g., 50%; see Elton, Gruber, Brown, and Goetzmann, 2010).

Sankaran and Patil (1999) studied "the problem of selecting portfolios which maximize the ratio of the average excess return to the standard deviation, among all those portfolios which comprise at most a pre-specified number, k, of securities. Under the assumptions of constant pairwise correlations and no short-selling, we argue that the simple ranking procedure of Elton, Gruber, and Padberg effectively solves the problem for all values of k, and that as a function of k, the optimal ratio increases at a decreasing rate. Using reasoning from graph theory, Tarrazo (2009) showed that the heuristic ri/stdi (individual security returns to standard deviation ratio) is the key variable responsible for the ranking of optimal weights. This means optimal portfolios can be calculated without using mathematical programming using, by adding securities one at a time (starting with that with the highest return) or subtracting those securities with negative weight (starting with the unrestricted, simultaneous equation solution to the problem; see Tarrazo, 2009, 2014). In this study, the evolution of n (preselected securities) to k (optimal securities) shares the non-proportional, power convergence rate exhibited by the portfolio optimal return-to-risk ratio during optimization (that is, rp/pstd, where "p" stands for portfolio).

5. The Search for the Heuristic k ≈ sqrt(n)

As noted earlier, the presence of patches with stable numbers for the number of optimal securities in critical, optimal portfolios (minimum-variance and tangent portfolios) obtained from any given n suggested the existence of a heuristic. Our first step was to review the objects and processes involved in the calculation of optimal portfolios. We found basic algebra concepts in $Z(x) = -1/2\ x'Ax + x'b$, a quadratic equation with its quadratic form ($x'Ax$). The optimization function is equivalent to the algebraic one, but one or two Lagrangians may be added to include restrictions for the required return or the sum of optimal weights. The important functional forms are the same as those in algebra. Finance research on the optimal number of securities also employs a variety of quadratic formulations and approximations, e.g., Markowitz (1959), Brennan (1975), and Shawky and Smith (2005). The statistical components in mean-variance portfolio theory are the calculation of two types of indicators (central tendency, mean; dispersion, variance) and the use of the normal distribution, which, unsurprisingly, can be expressed as an exponential function raise to a quadratic expression: f(x) = exp {ax^2 + bx + c}. The statistical component also includes some probability theory concerning convergence rates for estimates (sample to population) and for distribution themselves (e.g., central limits theorems). Interestingly, convergence formulas employ the square root of n, the number of observations.

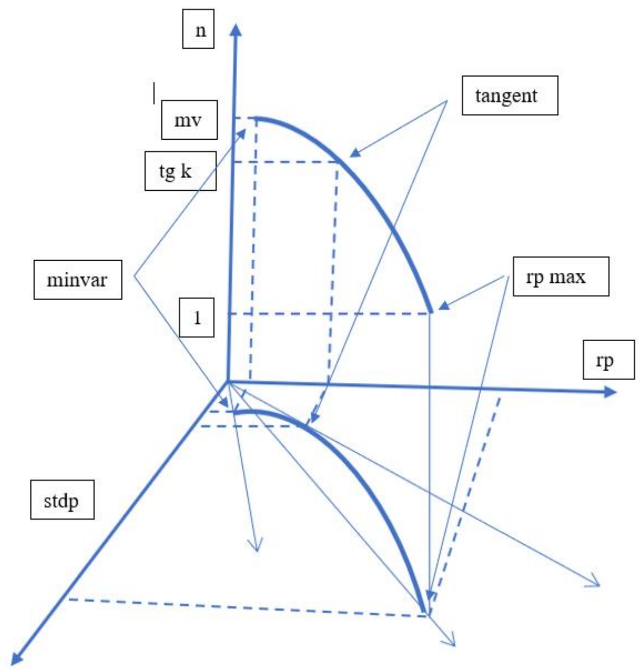


Figure 5. Efficient frontier portfolios, ratios and number of optimal securities

When we carried out portfolio optimizations, the tangent portfolio always had the largest return-to-risk ratio (rp/pstd), and it was easy to see that the minimum variance portfolio had a lower ratio and more securities, and the highest return portfolio (with 100% allocated to the security with the highest return) had both a lower return-to-risk ratio and only one security. That means that for n securities, the tangent portfolio will have k securities, and generally k is strictly less than n (i.e., k < n). We "cheated" by running prior optimization and selecting the best securities, but we still observed k < n. We also observe that the minimum variance portfolio always had more securities than the tangent but fewer than the total number of securities in the pre-optimization sample, that is n > mv > k (see Figure 2). As was true throughout the study, our analyses concerned only portfolios subject to non-negative weights restrictions.

Figure 5 shows the integration of the concepts used in portfolio optimization. We calculated the efficient frontier in the plane rp-pstd (width-length), and it included the three critical portfolios mentioned above. We added another axis (height) to represent the number of securities in each of those critical portfolios. We suspected there would be a relationship between those two curves, and, upon further inspection, we not only confirmed the relationship but found it to be quite special – both curves carried the same information; they were perfectly correlated in a special way. This numerical integration is shown in the Figure 6. The numbers correspond to Optimal Portfolio 10 in Figure 4. The numbers for each curve are at the top, graphically shown in a regular x-y chart (continuous line) to which we added a quadratic (polynomial order 2) approximation (dotted lines), and their equations appears to the right of the tittle. The R2 in each of the fitted lines was not surprising. It was the result of fitting a quadratic equation with three observations. What was remarkable was that the relation between the original curves was a conformal mapping. Despite their different units, they contained the same information; an appropriate change of coordinates shows their commonality. We show the perfect correlations between the information in each of the original curves in two ways: 1) using logarithms: we took the natural logarithms of the components in each curve, built their ratios, and computed the correlation and 2) using squares: we squared the components in each of the two original curves, built the proportions, and computed correlations. Using squares (norm = 2) is very common when applying a conformal mapping to a given matrix to compute eigenvalues and eigenvectors.

The two curves have similarity properties, both have a quadratic hump that is not situated in the arithmetic mean between the minimum and maximum values of the range (zero and n securities, respectively). It is closer to the minimum number in a nonlinear manner. And the statistics of convergence use square root of n, constituting a solid candidate for our fast and frugal rule-of-thumb:

The T-heuristics: For the tangent portfolio, given an initial securities, the optimal number of securities could be approximated by k = sqrt(n). One nonlinear, iteration down from n. The minimum-variance portfolio required two iterations: start with k = sqrt(n) and add another. That is, mv = 2*sqrt(n). The letter “t” indicates the transformations from the different, but related, coordinate systems the problem being solved inhabits.

At this point, we could assess the performance of the heuristic we derived. This assessment is presented in Exhibits VII and VIII. The information in Figure 7 is essentially a numeric reproduction of the information from the Figure 2 concerning optimal numbers of securities for each of the portfolio groupings, including some basic additional indicators (maximum and minimum values, and descriptive statistics). The actual performance of the heuristic is directly addressed in Figure 8, which contains a summary of the securities included in the optimal portfolios for each of the stock grouping during pre-crisis, crisis, and post-crisis periods.

The heuristic performed quite acceptably for the whole sample period, 01/31/2006-01/31/2011. It was right on target for the minimum-variance portfolio of the NDX index and deviated on only one security in three other occasions. It was only three securities away from the average for the BAF grouping, which included 131 securities, for a 2.30% error. The highest deviations consisted of a results five stocks away from the S&P100 (5 out of 93 stocks, 5.38%^), the NDX (5 out of 81, 6.17%), and BAF (5 out of 131, 3.81%). The heuristic underapproximated the mean for the tangent portfolio case.

The performance of the heuristic for the periods pre-, during, and post-crisis was quite good, with an underapproximation of the target during pre-crisis times. During and after the crisis, the most significant errors were two that occurred for the S&P100 (6 out of 93, 6.45%): one during the crisis for the tangent portfolio and another after the crisis for the minimum-variance portfolio.

Out of 32 analyses, the heuristic was perfectly correct in three cases. It was two or fewer securities away from the target in 50% of the cases and five or fewer securities away 85% of the time. These are quite modest errors given the Finance standards for regression errors and given the accuracy problems and ill-defined matrices used in portfolio optimization. In fact, it looks quite remarkable given that we were mapping ratios to integers and discretization is always tricky. Fortunately, Golub and Van Loan (1996) noted, “many of the matrices that arise from discretized elliptic PDE’s are symmetric positive definite” (p. 512). This means the integers may retain essential properties of their continuous source.

As with many other statistical tools (e.g., random number generating algorithms when simulating statistical distributions), the heuristic may be better evaluated in terms of existing in an area of wide variability rather than as squarely matching expected values.

After we applied the heuristic to the data and reviewed its logic, we tested the heuristic in a way we should have initially: using a simple x-y chart of a series of numbers (in this case, the first 20 integers) to their square roots. This chart is reproduced at the end of the Figure 8. But even given its simplicity, it provides a condensed summary of our findings. Although our variables of interest are expressed in their first order magnitude (e.g., returns, standard deviations, integers for the number of securities before and after the optimization), when we process risk we are working on a second order phenomenon. The results obtained from model optimizations are all related in a given, explainable manner. The x-y chart, y = x^2, represents the unit parabola, and all parabolas (e.g., the relationship between portfolio returns and portfolio variances) are affine transformations (translation plus linear mapping) of the unit case. Affine transformations are invariant; they preserve ratios, midpoints, and parallel lines, which are properties at play in the computation of coordinate equivalences between the original optimization and its corresponding integer expressions via conformal mappings. This we show graphically in Figure 5, and numerically in the Figure 6.

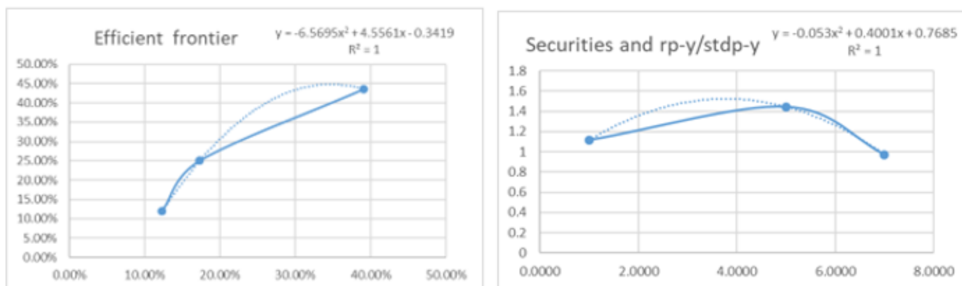
Perhaps the heuristics were just there waiting for someone to notice. Still, the conceptual integration was a worthwhile exercise. A preliminary explanation of the success in applying the square root rule in optimal portfolios is to think that we are observing simply a reflection of the many quadratically convergent processes we find in statistics. But there is more, much more to it than that. The “square root” heuristic may form part of the wider family of approximation around the power law, where some variables (authors, securities, people) receive a disproportionate share of a given collection of items – see, for example, Pareto's principle, Zipf's law, Lotka's law, Price's square root law. In addition, the square root rule is critical in both inventory and insurance –e.g., the economic ordering quantity (EOQ) follows a square root rule, in insurance and square root is used to approximate objective risk, etc. --See Appendix. These matters represent important areas for future research.

Efficient frontier:

stdp-y	rp-y
12.33%	12.00122%
17.33%	25.02842%
39.04%	43.55474%

Optimal securities and ratios:

n	rp-y/stdp-y
7.0000	0.973312
5.0000	1.444589
1.0000	1.115654



Conformal mapping check:

rpy/stdpy	ln ratios	n	ln securities
0.9733	**-0.39488**	7	**0.3364722**
1.4446	**0.258384**	5	**1.6094379**
1.1157		1	
		correlation	1

ratios^2	n^2	proportions squared ratios	securities
62.309698	49	**0.8962827**	**1.96**
69.520136	25	**8.5126361**	**25**
8.1666989	1		
		corr	1

Figure 6. Numerical integration efficient frontier and number of optimal securities

	DJIA (30)		S&P100 (93)		NDX (81)		BAF (131)	
	mv	ktg	mv	ktg	mv	ktg	mv	ktg
Whole: 1-72								
Average	9.75	5.68	18.15	13.65	17.32	13.43	19.13	15.58
Max	16.00	15.00	30.00	24.00	25.00	22.00	27.00	23.00
Min	5.00	2.00	10.00	5.00	13.00	8.00	15.00	7.00
Median	10.00	4.50	16.50	13.00	17.00	12.00	19.00	15.00
Average ^2	95.06	32.27	329.52	186.40	299.96	180.38	365.77	242.84
Variance	8.38	11.83	32.77	33.34	8.55	14.33	6.94	15.55
stdev	2.90	3.44	5.72	5.77	2.92	3.79	2.63	3.94
Pre-crisis 1-21								
Average	11.04	6.91	23.70	18.74	16.87	17.70	18.04	19.22
Max	14.00	14.00	27.00	24.00	23.00	22.00	22.00	23.00
Min	8.00	4.00	19.00	16.00	13.00	13.00	15.00	17.00
Median	11.00	7.00	23.00	19.00	16.00	18.00	18.00	19.00
Average ^2	121.96	47.79	561.48	351.16	284.58	313.14	325.57	369.31
Variance	1.61	7.30	5.26	3.67	10.90	2.82	3.43	2.43
stdev	1.27	2.70	2.29	1.92	3.30	1.68	1.85	1.56
Crisis: 24-42								
Average	12.50	7.80	20.10	14.80	19.65	12.80	22.05	14.60
Max	16.00	15.00	30.00	23.00	25.00	18.00	27.00	22.00
Min	9.00	2.00	13.00	5.00	14.00	8.00	19.00	7.00
Median	13.00	7.00	19.50	17.50	20.00	11.50	22.00	14.50
Average ^2	156.25	60.84	404.01	219.04	386.12	163.84	486.20	213.16
Variance	5.05	17.86	21.59	43.36	8.53	12.86	5.65	24.54
stdev	2.25	4.23	4.65	6.58	2.92	3.59	2.38	4.95
Post-crisis: 43-72								
Average	6.83	3.24	12.41	8.83	16.07	10.48	17.97	13.38
Max	9.00	5.00	16.00	13.00	19.00	13.00	21.00	19.00
Min	5.00	2.00	10.00	6.00	14.00	8.00	15.00	10.00
Median	7.00	3.00	12.00	8.00	16.00	10.00	18.00	13.00
Average ^2	46.62	10.51	154.10	77.93	258.21	109.89	322.76	179.01
Variance	0.97	1.01	2.38	5.25	1.24	1.08	2.45	3.75
stdev	0.99	1.01	1.54	2.29	1.11	1.04	1.56	1.94

Figure 7. Optimal securities in efficient portfolios

	DJIA (30)		S&P100 (93)		NDX (81)		BAF (131)	
	mv	ktg	mv	ktg	mv	ktg	mv	ktg
Whole: 1-72								
Average	9.75	5.680556	18.15277778	13.65278	17.31944	13.43056	19.125	15.58333
Pre-crisis 1-21								
Average	11.0434783	6.913043	23.69565217	18.73913	16.86957	17.69565	18.04347826	19.21739
Crisis: 24-42								
Average	12.5	7.8	20.1	14.8	19.65	12.8	22.05	14.6
Post-crisis: 43-72								
Average	6.82758621	3.241379	12.4137931	8.827586	16.06897	10.48276	17.96551724	13.37931
Heuristic	**2*sqrt(n)**	**sqrt(n)**	**2*sqrt(n)**	**sqrt(n)**	**2*sqrt(n)**	**sqrt(n)**	**2*sqrt(n)**	**sqrt(n)**
	10.9544512	5.477226	19.28730152	9.643651	18	9	22.89104628	11.44552
Whole: 1-72	1	-1	1	-5	0	-5	3	-5
Pre-crisis 1-21	-1	-2	-5	-10	1	-9	4	-8
Crisis: 24-42	-2	-3	-1	-6	-2	-4	0	-4
Post-crisis: 43-72	4	2	6	0	1	-2	4	-2

Difference	Out of 32 Occurrences	%	Cumulative %
0	3	9.38%	9.38%
1	7	21.88%	31.25%
2	6	18.75%	50.00%
3	2	6.25%	56.25%
4	5	15.63%	71.88%
5	4	12.50%	84.38%
6	2	6.25%	90.63%
7	0	0.00%	90.63%
8	1	3.13%	93.75%
9	1	3.13%	96.88%
10	1	3.13%	100.00%
	32		

A quadratic relationship: y = sqrt (n)

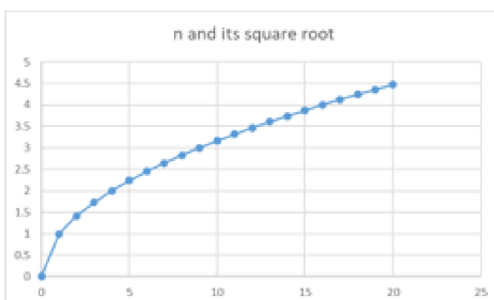


Figure 8. Heuristic performance

6. Concluding Comments

In this study, we present two heuristics, which we call “t-heuristics” from their square-roo**t** origins: For the tangent portfolio, given “n” initial securities, the optimal number of securities may be approximated as k = sqrt(n), one nonlinear iteration down from n. The minimum-variance portfolio requires two iterations; start with one k = sqrt(n) and add another. That is, mv = 2*sqrt(n). The letter “t” indicates the transformations from the different, but related, coordinate systems the problem being solved inhabits.

Our findings originated from accumulated experience optimizing hundreds of small portfolios and evaluating their value for actual investing and from the systematic empirical research in this study. We present material on the plausibility and theoretical support for the heuristics and their consistency from numerical and mathematical (i.e., statistical theory, conformal mapping) viewpoints.

The concept that the number for the optimal securities could be the (approximate) square root of n, the initial number of securities, suggested itself in a sea of quadratic constructs. Like every heuristic worthy of that name, it should be helpful. First, it originates in what still might be the best tool we have to put together portfolios – the mean-variance model. Admittedly, this model is only a crude approximation, but as Samuelson (1967) put it, "…in practice, where crude approximations may be better than none, the 2-moment models may be found to have pragmatic usefulness" (p. 12). In addition, awareness of the quadratic relationship between the pre- and post-optimization number of securities in an investment portfolio opened other vistas. For example, how many securities should be analyzed? Studying about 20-30 stocks, the equivalent to focusing on the DJIA, to identify a small portfolio of five to six securities looks very feasible and very attractive, especially when looking at their return-to-risk ratios – nothing short of spectacular in several of them. In a way, and somewhat independently of the value of the heuristic, the analysis presented highlights the suitability of small portfolios.

The heuristics presented may also prove to be useful in other contexts where historical information is lacking, for example, in venture capital investing, where sometimes over a hundred proposals yield ten or even fewer investable projects. The fact that a second order process provides a square root heuristic suggests pursuing a third order process logic (average, variance, asymmetry) might provide a cube root heuristic.

Jagannathan and Ma (2003) wrote and very informative study on the number of securities in optimal portfolios; there we read, "A striking feature of minimum variance portfolios constructed subject to the restriction that the portfolio weights should be nonnegative is that investments is spread over only a few stocks. The minimum variance portfolio of a 500-stock universe has between 24 and 40 stocks, depending on which covariance matrix estimator was used…" (p. 1677). Somehow, we were not surprised; the square root of 500 is 22.36, and twice that is 44.72.

References

Åstebro, T., & Elhedhli, S. (2006). The Effectiveness of Simple Decision Heuristics: Forecasting Commercial Success for Early-Stage Ventures. *Management Science, 52*(3), 395-409. https://doi.org/10.1287/mnsc.1050.0468

Baker, H. K., & Nofsinger, J. (2002). Psychological Biases of Investors. *Financial Services Review, 11*(2), 97-116. https://doi.org/10.1002/cb.1644

Ball, M. (2011). Heuristics Based on Mathematical Programming. *Surveys in Operations Research and Management Science, 16*, 21-38. https://doi.org/10.1016/j.sorms.2010.07.001

Beck, K., Perfect, S., & Peterson, P. (May 1996). The Role of Alternative Methodology on the Relation Between Portfolio Size and Diversification. *The Financial Review, 31*(2), 381-406. https://doi.org/10.1111/j.1540-6288.1996.tb00878.x

Benjelloun, H. (2010). Evans and Archer -Forty Years Later. *Investment Management and Financial Innovations, 7*(1), 98-104. No DOI located.

Bird, R, & Tippett. (February 1986). Note—Naive Diversification and Portfolio Risk—A Note. *Management Science, 32*(2), 139-256. https://doi.org/10.1287/mnsc.32.2.244

Brennan, M. (September, 1975). The Optimal Number of Securities in a Risky Asset Portfolio When There are Fixed Costs of Transacting: Theory and Some Empirical Results. *The Journal of Financial and Quantitative Analysis, 10*(3), 483-496. Administration Stable URL: https://doi.org/10.2307/2330492

Britten-Jones, M. (December 2002). The Sampling Error in Estimates of Mean-Variance Efficient Portfolio Weights. *The Journal of Finance, 54*(2), 655-671. https://doi.org/10.1111/0022-1082.00120

Boscaljon, B., Filbeck, G., & Ho, C. (Fall 2005). How Many Stocks Are Required for a Well-Diversified Portfolio? *Advances in Financial Education, 3*, 60-71. No DOI located.

Bronshtein, I., & Semendyayev, K. (1985). *Handbook of Mathematics*. Van Nostrand Reinhold Company. https://doi.org/10.1007/978-3-662-21982-9

Chhavra, A. (Spring 2005). Beyond Markowitz: A Comprehensive Wealth Allocation Framework for Individual Investors. *The Journal of Wealth Management*, 8-34. https://doi.org/10.3905/jwm.2005.470606

Constantinides, G. M., & Malliaris, A.G. (1995). Portfolio Theory. Chapter 1 in Robert Jarrow et al., *Handbook in OR & MS, 9.* Elsevier Science, New York. https://doi.org/10.1016/S0927-0507(05)80045-3

De Wit, & D. P. M. (1998). Naive diversification. *Financial Analysts Journal, 54*(4), 95-100.

https://doi.org/10.2469/faj.v54.n4.2201

Domian, D., Louton, D., & Racine, M. (November 2007). Diversification in Portfolios of Individual Stocks: 100 Stocks Are Not Enough. *The Financial Review, 42*(4), 557-570. https://doi.org/10.1111/j.1540-6288.2007.00183.x

Elton, E., Gruber, M., Brown, S., & Goetzmann, W. (2010). *Modern Portfolio Theory and Investment Analysis*. 8th. ed., Wiley, New York. No DOI located.

Elton, E., & Gruber, M. (February 1977). Risk Reduction and Portfolio Size: An Analytical Solution. *The Journal of Business, 50*(4), 415-37. https://doi.org/10.1086/295964

Evans, J., & Archer, S. (December 1968). Diversification and the Reduction of Dispersion: An Empirical Analysis. *The Journal of Finance, 23*(5), 761-767. https://doi.org/10.2307/2325905

Fabozzi, F., Gupta, F., & Markowitz, H. (Fall 2002). The Legacy of Modern Portfolio Theory. *The Journal of Investing*, *1*(3), 7-22. https://doi.org/10.3905/joi.2002.319510

Fama, E. F. (1965). Portfolio Analysis in a Stable Paretian Market. *Management Science, 11*(3), 404-419. https://doi.org/10.1287/mnsc.11.3.404

Fielitz, B. (Winter, 1974). Indirect versus Direct Diversification. *Financial Management*, *3*(4), 54-62. https://doi.org/10.2307/3664930

Fisher, L., & Lorie, J. (1970). Some Studies of Variability of Returns on Investments in Common Stocks. *The Journal of Business, 43*(2), 99-134. https://doi.org/10.1086/295259

Fonseca Costa, D., De Melo Carvalho, F., & de Melo Moreira, B. (March 2018). Behavioral Economics and Behavioral Finance: A Bibliometric Analysis of the Scientific Fields. *Journal of Economic Surveys, 33*(1). Free Access Article: https://doi.org/10.1111/joes.12262

Geoffrion, A. (1976 November). The Purpose of Mathematical Programming Is Insight, Not Numbers. *Interfaces, 7*(1), Part 1 of Two, 81-92. https://doi.org/10.21236/ADA030006

Goetzmann, W. N., & A. Kumar. (2008). Equity Portfolio Diversification. *Review of Finance, 12*(3), 433-463. https://doi.org/10.1093/rof/rfn005

Goldman, B. (May 1979). Anti-Diversification or Optimal Programs for Infrequently Revised Portfolios. *The Journal of Finance, XXXIV*(2), 505-516. https://doi.org/10.2307/2326993

Golub, G., & Van Loan, C. (1996). *Matrix computations* (3rd ed.). Baltimore and London: The Johns Hopkins University Press. No DOI located.

Hogarth, R., & Karelia, N. (Nov., 1976). Simple Models for Multi-Attribute Choice with Many Alternatives: When It Does and Does Not Pay to Face Trade-offs with Binary Attributes. *Management Science, 2005. Interfaces,* 7(1), Part 1 of Two, 1733-1902. https://doi.org/10.1287/mnsc.1050.0448

Ivkovick, Z., Sialm, C., & Weisbenner, S. (September 2008). Portfolio Concentration and the Performance of Individual Investors. *Journal of Financial and Quantitative Analysis, 43*(3), 613-656. https://doi.org/10.1017/S0022109000004233

Jagannathan, R., & Ma, T. (August 2003). Risk Reduction in Large Portfolios: Why Imposing the Wrong Constraints Helps. *The Journal of Finance*, *58*(4), 1651-1683. https://doi.org/10.3386/w8922

Jacob, N. (1974). A Limited-Diversification Portfolio Selection Model for the Small Investor. *Journal of Finance, 29*(3), 847-856. https://doi.org/10.2307/2978596

Jennings, E. H. (1971). An Empirical Analysis of Some Aspects of Common Stock Diversification. *Journal of Financial and Quantitative Analysis*, *6*(2), 797-813. https://doi.org/10.2307/2329715

Jobson, J., & Korkie, R. (June 1983). Statistical Inference in a Two-Parameter Portfolio Theory with Multiple Regression Software. *Journal of Financial and Quantitative Analysis, 18*(2), 189-197. https://doi.org/10.2307/2330917

Kryzanowski, L., & Singh. (October 2010). Should Minimum Portfolio Sizes Be Prescribed for Achieving Sufficiently Well-Diversified Equity Portfolios? *Frontiers in Finance and Economics, 7*(2), 1-37. DOI not located.

Mao, J. (December, 1970). *Essentials of Portfolio Diversification Strategy, 25*(5), 1109-1121. https://doi.org/10.2307/2325582

Markowitz, H. (1959). *Portfolio Selection, Efficient Diversification of Investments*. New York, John Wiley and Sons.

DOI not located.

Markowitz, H. (1952). Portfolio Selection. *Journal of Finance, VII*(1), 77-91. https://doi.org/10.2307/2975974

Martellini, L., & Branko Urošević. (2006). Static Mean-Variance Analysis with Uncertain Time Horizon. *Management Science, 52*(6), 955-964. https://doi.org/10.1287/mnsc.1060.0507

Newbould, G., & Poon, P. (Fall 1993). The Minimum Number of Stocks Needed for Diversification. *Financial Practice and Education, 3*(2), 85-87. DOI not located.

Rekenthaler, J. (September 1999). Strategic Asset Allocation: Make Love, Not War. *Journal of Financial Planning, 12*(8), 32-34. DOI not located.

Riepe, M. (May 2002). Diversification, Vegetables and Bill Gates. *Journal of Financial Planning*, 36-38. DOI not located.

Samuelson, P. (1967). A General Proof that Diversification Pays. *Journal of Financial and Quantitative Analysis, 2*(1), 1-13. https://doi.org/10.2307/2329779

Sankaran, J., & Patil, A. (1999). On the optimal selection of portfolios under limited diversification. *Journal of Banking & Finance, 23*, 1655-1666. https://doi.org/10.1016/S0378-4266(99)00023-0

Sengupta, J., & Sfeir, R. (1985). Tests of Efficiency of Limited Diversification Portfolios. *Applied Economics, 17*(6), 933-945. https://doi.org/10.1080/00036848500000058

Shawky, H., & Smith, D. (2005). Optimal Number of Stocks Holdings in Mutual Fund Portfolios based on Market Performance. *The Financial Review, 40*, 481-495. https://doi.org/10.1016/j.ejor.2013.01.024

Silver, E. A. (September, 2004). An Overview of Heuristic Solution Methods. *The Journal of the Operational Research Society, 55*(9), 936-956. https://doi.org/10.1057/palgrave.jors.2601758

Simon, H., & Newell, A. (January-February, 1958). Heuristic Problem Solving: The Next Advance in Operations Research. *Operations Research, 6*(1), 1-10. https://doi.org/10.1287/opre.6.1.1

Smith, J., & Winterfeldt, D. (May 2004). *Decision Analysis in Management Science, 50*(5), 561-574. https://doi.org/10.1287/mnsc.1040.0243

Statman, M. (1987). How Many Stocks Make a Diversified Portfolio. *Journal of Financial and Quantitative Analysis, 22*(3), 353-363. https://doi.org/10.2307/2330969

Stigler, G., & Becker, G. (March 1977). De Gustibus Non Est Disputandum. *The American Economic Review, 67*(2), 76-90. http://www.jstor.org/stable/1807222

Tarrazo, M. (Winter & Spring 2018). Mean-Variance under Stress -Ten years of Portfolio Optimizations: 2001-2011. *Quarterly Journal of Finance and Accounting, 50*(1 & 2), 91-148. DOI not located.

Tarrazo, M. (2014). Portfolio Optimization without Programming –Pedagogical and Practical Implications. *Advances in Financial Education, 12*, 90-122. DOI not located.

Tarrazo, M. (2009). Identifying Securities to Buy: The Heuristic ri/stdi. *Research in Finance, 25*, 229-268. https://doi.org/10.1108/S0196-3821(2009)0000025011

Tarrazo, M. (March 2024). *On Fibonacci's Square Roots Solutions Method: Quadratic Equations, Proportionality and Scaling*. School of Management, University of San Francisco. Email: tarrazom@usfca.edu

Vissing-Jorgensen, A. (2003). Perspectives On Behavioral Finance: Does "Irrationality" Disappear with Wealth? Evidence From Expectations and Actions. Kellogg School of Management, Northwestern University, NBER and CEPR June 2, 2003. Chapter in NBER book *NBER Macroeconomics Annual 2003, 18*(2004), Mark Gertler and Kenneth Rogoff, editors (p. 139 - 208) Conference held April 4-5, 2003. Published in July 2004 by The MIT Press in NBER Book Series NBER Macroeconomics Annual; https://doi.org/10.1086/ma.18.3585242

Wagner, W., & Lau, S. (November- December, 1971). The Effect of Diversification on Risk. *Financial Analysts Journal, 27*(6), 48-53. https://doi.org/10.2469/faj.v27.n5.48

Woerheide, W., & Person, D. (1992-93). An Index of Portfolio Diversification. *Financial Services Review, 2*(2), 73-85. https://doi.org/10.1016/1057-0810(92)90003-U

Zaimovic, A., Adna O., & Arnaut-Berilo, A. (2021). How Many Stocks Are Sufficient for Equity Portfolio Diversification? A Review of the Literature. *Journal of Risk and Financial Management, 14*, 551. https://doi.org/10.3390/jrfm14110551

Appendix

An Optimal Inventory Understanding of Portfolio Optimization

This appendix clarifies the point raised in the last section of the study about the foundation of the heuristic k = sqrt(n) in the theory of optimal inventories.

It is fair to think of the Economic Ordering Quantity (EOQ) model as one of the most robust foundations in optimal inventory research and practice. It is a foundation not only for inventories but also for organizing production, deliveries, and processes requiring scheduling. In fact, the article first describing both the general ideas and specifics of what would be later known as EOQ was titled "How many parts to make at once?", written by Ford Harris in 1913. R. H. Wilson extended Harris' analysis and added a distinctive industrial engineering approach that he described as a scientific routine for stock control. Wilson's 1934 study included interesting, pre-computer, decision-making aids such as stock record cards, ordering point calculating wheels with movable overlapping disk, and ordering point tabular representations of data. The EOQ became part of the operations-research, quantitative management toolbox. In the 1950s, it was used by William Baumol to develop a model for transactions demand for cash which, in turn, helped formulate an influential theory regarding the relationship between the rate of interest and the demand for cash by James Tobin, and a whole model dedicated to the demand for money by firms by Merton Miller and Daniel Orr.

We searched both the practitioner and the research literature and found some interesting extensions of the basic EOQ such as, for example, the following: The EQO variant known as the Economic Production Quantity (EPQ); the (S,s) model; McDaniel's inverted EOQ approach, where the inventory is replenished at a constant rate until it reaches a given value;

Brennan (1975) where transaction costs limit the number of securities in a portfolio following a quadratic polynomial; and Stigler and Becker (1977) where the optimal number for searches follows the EOQ, which is derived from a quadratic equation. The literature is immense and cover many areas. The good news is that, for the purposes of this study, the EOQ entry in Wikipedia may suffice as an introduction to the model: https://en.wikipedia.org/wiki/Economic_production_quantity

Let's see how the EOQ works; let's first introduce the variables used by the model with some numbers: level of sales (S = 200,000), per unit costs of carrying the inventory (C = 1.50), and some fixed costs of placing an order to replenish the inventory (P = 100). The optimal inventory, Q = EOQ = (p S 2/c)^ ½ = 5,163 units, will minimize the total (ordering and carrying) cost of inventory. The expression for the Ordering Costs is, OC = [S/Q] * P .The expression for the Carrying Costs is, CC = ½ Q * C . The expression for the EOQ can be found by taking first order derivatives in [OC + CC] with respect to Q, and making the resulting equation equal to zero. However, it is much easier just to first equate OC to CC, extract Q^2, and then calculate the square root.

We can rewrite the solution found as follows: EOQ = square root (2 S **P/C**), where we see that the term square root (2 S) is modified by a composite factor, a ratio of something magnifying the optimal amount of Q --that would be P-- though the ordering pattern, and something diminishing the optimal amount Q --that would be C-- acting through the carrying costs. If we look at the transaction costs, we even see more clearly the magnifying effect of P on S/Q.

This is very important: why not Q to S right away? –Because, implicit in the model is the fact that while Q is immediately wanted/needed, S is not. There is something that makes S not immediately desirable/needed. This is exactly what happens when an investor puts together a portfolio of stocks. S/he examines n stocks and the corresponding firms, and then gathers data on their returns (statistical averages), and risk indicators (statistical variances and covariances). The larger set of securities, of size n, is of definite interest, but there is something that makes the smaller set, of k securities immediately purchasable, that is revealed by a quadratic, mean-variance optimization.

It turns out that in statistical analysis, the coefficient of variation, cv = standard deviation/average –the average usually represents the expected value, in sampling, regression, or forecasting analyses. And it also happens that portfolios are evaluated in terms of their return-to-risk ratio (rp/stdp), stdp representing the square root of the variance of the portfolio. We can easily integrate both the EOQ and MV portfolio optimization by the following operations: replacing S by n, Q, by k, and the ratio P/C, by stdp/rp.

Figure 9 shows an optimal portfolio, and its return (rp =), and risk (stdp =). This means that the optimal number of securities, k, would be explained/approximated by: k = square root (2*n*stdp/rp) = square root (2 * 10 * 0.543294) = 6 , approximately.

In the middle of the Figure 9, we also show that alternative heuristic rules, such as sqrt(n), or sqrt(2n) can be combined. For example, averaging the EOQ and the sqrt(2n) heuristic provide the exact number of securities in the optimal portfolio: INT((EOQ + sqrt(2n))/2) = 5.

We can also apply this model to stock market indices. For example, we can read the following with respect to the risk-return statistics of the S&P 500: "Since its inception in 1926, the index's compound annual growth rate—including dividends—has been approximately 9.8% (6% after inflation), with the standard deviation of the return over the same time period being 20.81%." S&P 500 - Wikipedia https://en.wikipedia.org/wiki/S%26P_500

The lower part of Figure 9 shows what happens if we calculate the optimal portfolio of securities without EOQ-MV model. The top line shows the number of initial securities (n), and the last line shows the approximated number of securities in the optimal portfolio (k). It can be read as if asking, "If I start with 500 securities, how many may I end up with in an optimal portfolio?" It can also be read the other way, "If I would like to have a portfolio of k securities, how many should I examine and analyze? The answer, start with 20 if you want to end up with a portfolio of 9, or 10 if you want to end up with a portfolio of 6 is very reasonable in terms of search and follow up cost and effort.

As we see, the integration of the EOQ and mean-variance models produces a very plausible approximation that complements other approximations of interests k = sqrt(n), and k = sqrt(2 n). The information presented suggests additional research on how further theoretical foundations for the heuristic presented could likely be found in the analysis by Brennan (1975), and Stigler and Becker (1977). In the context of this study, this appendix has presented as much information as needed to strengthen the foundations of a square root (i.e., quadratic) heuristic rule for portfolio construction.

	AAPL	*ABM*	*ADSK*	*ACN*	*ADBE*	*ADPT*	*ALTR*	*AMAT*	*AMD*	*AMH*
AAPL	0.017352	0.001638	0.003129	0.00139	0.008468	0.004539	0.001507	0.001174839	0.002422	0.002849
ABM	0.001638	0.003542	-0.00043	0.001513	-4.1E-05	0.000544	-0.00154	0.000166489	-0.0003	0.001427
ADSK	0.003129	-0.00043	0.01433	0.001614	0.006032	0.003914	0.001794	0.007328882	0.005162	0.004434
ACN	0.00139	0.001513	0.001614	0.011077	-0.00164	0.003048	0.00288	0.002126306	0.002902	0.001987
ADBE	0.008468	-4.1E-05	0.006032	-0.00164	0.025269	0.002688	-0.0011	0.006095602	0.001903	0.005588
ADPT	0.004539	0.000544	0.003914	0.003048	0.002688	0.014173	0.005062	0.005021787	0.001993	0.0017
ALTR	0.001507	-0.00154	0.001794	0.00288	-0.0011	0.005062	0.022801	0.009416929	0.006628	0.002374
AMAT	0.001175	0.000166	0.007329	0.002126	0.006096	0.005022	0.009417	0.015693199	0.006542	0.00502
AMD	0.002422	-0.0003	0.005162	0.002902	0.001903	0.001993	0.006628	0.006541982	0.022408	-0.00012
AMH	0.002849	0.001427	0.004434	0.001987	0.005588	0.0017	0.002374	0.005020367	-0.00012	0.019216

n		*ri*
1	AAPL	-0.01183
2	ABM	**0.017287**
3	ADSK	0.010489
4	ACN	0.00944
5	ADBE	**0.018137**
6	ADPT	0.034321
7	ALTR	0.053437
8	AMAT	0.04449
9	AMD	0.019065
10	AMH	0.004993
n= 10		

k	*Optimal*	*weights*
1	ABM	0.516614
2	ADBE	0.036169
3	ADPT	0.105439
4	ALTR	0.221221
5	AMAT	0.120557
k = 5		

MV optimization	
rp	0.030391
stdp	0.055938
varp	0.003129
rp/stdp	0.543294
2*stdp/rp*10	36.81246
sqrt(2*stdp/rp*10)=**EOQ**	6.067327
sqrt(n)	3.162278
sqrt(2n)	4.472136
INT((EOQ + sqrt(2n))/2)	5

n	500	100	20	10
rp	9.8	9.8	9.8	9.8
stdp	20.81	20.81	20.81	20.81
2 *n * stdp/rp	2123.469	424.6939	84.93878	42.46939
EOQ-k	46	20	9	6

Figure 9.