Roy's (1952) Revisited in Today's Investing Contexts

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Received: June 14, 2022 Accepted: June 28, 2022 Online Published: June 30, 2022

doi:10.5430/afr.v11n3p14 URL: https://doi.org/10.5430/afr.v11n3p14

Abstract

A. D. Roy's "Safety First and the Holding of Assets" (Econometrica, 1952), is not only highly regarded but also considered one of the cornerstones of portfolio theory. Roy (1952) provided a modern approach to wealth multiples, information-based risk management, which included a re-evaluation of the role of cash reserves deployed in a multiperiod setting. These ideas and the work derived from them have contributed to progress in the modeling of inventory management, money demand, and portfolio insurance (value-at-risk being a modern incarnation of Roy's safety-first principle). However, not a single existing study in the finance literature employed a strict application of Roy's investing approach, directly as it was laid out in his original contribution. This is exactly the objective of our study. Furthermore, at each step of the application of Roy's model, we compare it to the corresponding step in Markowitz's (1952, 1959) mean-variance model. We find that the most forward-looking elements in Roy's model, mentioned above, decouple revenues from risk. Specifically, the variance-covariance matrix, a mainstay of mean-variance analysis, gets dislodged from the computations, which prevents the calculation of the efficient frontier, as noted by Markowitz (1987, p. 37). We show that revenues-risk integration takes place in Roy's model at the level of expectations, as in expert assessments, beyond the purely descriptive statistics of the mean-variance model. What appear as limitations from the standard mean-variance model viewpoint are, therefore, turned into advantages in certain contexts such as venture capital, angel investing, and private equity.

Keywords: investment analysis, portfolio theory, mean-variance, safety-first, startup investing, venture capital

1. Introduction

In this study, we revisit A. D. Roy's (1952) "Safety First and the Holding of Assets." This is one of three studies that introduced portfolio analysis into modern finance. Markowitz's (1952), which appeared in the same year, became the standard reference for portfolio analysis, and the more specific term of "mean-variance" analysis became synonymous with his name. Tobin (1958) integrated mean-variance into the wider settings of liquidity and money demand. A few years later during the 1960s, portfolio theory became a commonly used support and foundation not only for equity markets but also for financial markets, for integrating real and financial assets and, and for theoretical and applied decision analysis in general. Portfolio theory is now at the heart of modern finance, supporting fundamental concepts (e.g., arbitrage, market efficiency), facilitating financial innovation (e.g., derivatives, pooled investing), and informing practical decisions in investing (e.g., mutual funds) that affect each one of us.

Revisiting Roy's work is quite timely and, perhaps somewhat surprisingly, provides much food for financial thought. More than ever, today's readers can observe an undeniable practical investing bent in Roy's approach. In addition, the approach seems especially well suited for modern individual investors, who have wider access to direct investing in venues such as angel investing. However, Roy's expository strategy seems to have prevented many readers from plunging into his study with the same enthusiasm they applied to Markowitz's early contributions (1959, 1952). Markowitz (1952) kept discussion of the mathematical apparatus and probability theory to a minimum and favored graphical analysis at each turn in the road. This strategy served him well, even when dwelling on how to optimize a portfolio using what was to be known as "the critical line algorithm" – not an easy topic at all (see Kwan, 2007, for a clear exposition of the algorithm using spreadsheets). In contrast, Roy immediately set up a probability theorem, established a tricky correlation, and introduced conics to find analytical solutions, using symbolic analysis at each step. Of course, the study appeared in the quantitative scholarly journal Econometrica, and many of its readers may have read and understood it easily. But the broader public has not viewed the study as a way to improve day-to-day investing as they have Markowitz's (1952). His approach became "the bridge" and standard of reference in the literature: "Markowitz (1952) marks the beginning of modern portfolio theory, where for the first time, the problem of portfolio selection is clearly formulated and solved" (Constantinides and Malliaris, 1995, p. 2). Interested readers may consult Markowitz (1999) and Rubinstein (2002, 2006) for details concerning the evolution and impact of portfolio theory in modern finance.

Roy's achievements were undoubtably considerable. As Markowitz has noted, "had his [Roy] objective been to trace out mean-variance efficient sets we would have called it Roy's model" (Markowitz, 1987, p. 37). Roy (1952) deployed a mathematical analysis of portfolio optimization that would not to be picked up, until nearly 20 years later (see Roll, 1977, p. 177 and ss.). In the many years after publication, Roy's (1952 and 1956) insurance-like treatment of portfolio investments (safety-first) has motivated much research (value-at-risk, shortfall analysis). Still, the literature lacks a literal application of Roy's (1952) model, and it is often employed with modifications (e.g., percentage risk-free return losses instead of dollar amount; see for example Milevsky, 2002).

We recently took another look at Roy (1952) because we felt that Roy's approach and model have not yet found the best context for practical deployment. A quick overview reading of Roy (1952) reveals at least the following elements worth exploring:

1. A practical and interesting model for individual investors – values, small portfolios, and cash both as part of the money demand of the individual and as a reserve for taking advantage of equity opportunities. This reserve is part of Roy's often quoted but rarely deployed "safety first" principle.
2. Investing tools that are helpful for students and researchers.
3. A model that could be of some interest for entrepreneurial and venture capital – dollar amounts, wealth ratios, and small portfolios.
4. A bridge between quantitative and qualitative portfolio analyses.
5. Finally, effective use of Roy's (1952) approach implies focusing on prices and wealth multiples in a multiperiod context, which could be a better option than simply using returns.

That is no small hoard of treasure to make use of. The last point, even in itself, would justify further re-reading of Roy (1952). As noted earlier, one cannot find examples of implementations of his model as of his hypothetical investor (small, individual) would use it. That application is the objective of the first part of our study. There, we revisited the variables and his model, which triggered a clarification concerning correlation-based analysis. Finally, we used leverage-portfolios construction techniques implementing Roy's safety-first principle. Points (1) and (5) above bring to mind specialized investors (angels, venture capitalists, private equity) in specialized fields (startups, privately owned established firms). We close our study with brief concluding comments.

Despite our enthusiasm, we must acknowledge that Roy (1952) does not offer an easy read, even by today's standards. Bringing his contribution to a modern and wide audience benefits from implementing his model in parallel to the most straightforward mean-variance model one can possibly put together. Part of the popularity of mean-variance analysis has been its intuitive appeal, but its development also offers examples of unnecessary complications, which compounded the challenges presented by some of its ambiguous areas (how to preselect securities, frequency and length of samples, size of the pre-optimization set, planning period, revisions and trading, etc.). Unnecessary mathematical complications have also hurt the appeal of Markowitz's mean-variance model as when using "inessential generality," as noted by Steinbach (2001).

Roy (1952) may attract professionals from different backgrounds (finance, probability and statistics, mathematical modeling) and interests (theoretical, applied investing). Throughout our study, we present the viewpoint of sophisticated individual investors (or specialized investors), perhaps a fine business graduate, who picked up Roy's article to see if they could enhance their practical investing strategies. Instead of his combination of mathematical probability theory plus multivariate calculus, we used far more inviting algebra-based tools and accomplished the objective using only one formula.

Roy (1952) may appear as an eminent researcher that may have built an impressive line of follow-up research after his initial article, but that does not seem to have been the case except for his research on safety-first related concepts (expected gain confidence limit, portfolio shortfall constraints, value-at-risk, and so on). However, in his career, Roy seems to have followed policy and economic decision-making, which requires some professional gifts for applied practice, and those gifts are discernible in his inspiring study.

The initial bibliography and the references selected were primarily concerned with providing solid foundations of the analysis presented to a diverse audience of researchers, and practitioners in both finance and entrepreneurship. At the suggestion of a referee, we added these additional references to better integrate our analysis to recent research and to prepare for future work as well: Scherer (2013) notes that small portfolios may not necessarily be undiversified. Lassance (2022) shows that mean-variance theory provides reasonable numbers in the presence of non-Gaussian returns. Zweig (2022), writing in the context of mutual and exchange traded funds, illustrates why "bigger is not better." Lo and Foerster (2022), after reviewing major investment theories, suggest that optimal portfolios may be unique to each individual investor's situation, characteristics, and experiences. A view that is fully compatible with the work we present here.

Roy (1952)

1. Understand Roy's objectives and the way he sets the calculations (variables, meaning, etc.)
2. Understand his modeling set up – some but not all of which comes from probability theory – and how it yields an optimizing solution. There are two subproblems here: 2.a) seeing thru the probability theory relationships and 2.b) the computation of actual numbers.
3. Understand the workings of what is normally referred to as levered portfolios. These are the constructs Roy set up, in multiperiod dollar denominated accounts, with their associated probability indicators (mean and variance), presumably via a correlation matrix. The conventional mean-variance set up integrates single period-rates of returns with their associated probability indicators (mean and variance) with a variance-covariance matrix.
4. Critically comparing and evaluating the numbers provided by both Roy's and Markowitz's models in the proper context, which is not merely mathematical correctness but offers the possibility of improving investing practice.

1.1 Roy's Objectives and Variables

Roy designed his model for individual investors with relatively modest means, limited knowledge, and only a few chances to get it right. For them, investing is not a matter of averages, "… the ordinary man has to consider the possible outcomes of a given course of action on one occasion only and the average (or expected) outcome, if this conduct were repeated a large number of times under similar conditions, is irrelevant" (ibid, p. 431).

In his model, investors allocate a given amount (we used $100,000 for expository convenience and easy comparisons of dollar-denominated wealth and allocations with returns and percentage values) over a few investments assets not limited to common stocks (we uses four common stocks). The individual looks at the present price of those assets (p1t, p2t, p3t, p4t), and estimates their level at the end of his/her investment engagement (p1T, p2T, p3T, p4T), which provides expected wealth ratios or multiples. The investing horizon may consist of periods comprising several years each.

For this small, individual, household investor, the whole process takes place in a hostile environment and with expected hardship:

"A valid objection to much economic theory is that it is set against a background of ease and safety. To dispel this artificial sense of security, theory should take account of the often-close resemblance between economic life and navigation in poorly charted waters or manoeuvres in a hostile jungle. Decisions taken in practice are less concerned with whether a little more of this or of that will yield the largest net increase in satisfaction than with avoiding known rocks of uncertain position or with deploying forces so that, if there is an ambush round the next corner, total disaster is avoided. If economic survival is always taken for granted, the rules of behaviour applicable in an uncertain and ruthless world cannot be discovered." (1952, p. 436)

Roy argues these small investors to distribute their resources beyond any one asset (this suggests hedging rather than diversification), but also avoid financial disaster:

"Now let us attack the problem of how should distribute our sources between different assets, given certain expectations about the future... It is sufficient that our resources should not end in terms of any one asset. We wish to ensure that at the reckoned of a given period our resources do not fall to or below an amount d also in real terms." (1952, p. 435-6)

"In the economic world, disasters may occur if an individual makes a net loss as the result of some activity, if his resources are eroded by the process of inflation to, say, 70 per cent of their former worth, or if his income is less than what he would almost certainly obtain in some other occupation. For large numbers of people some such idea of a disaster exists, and the principle of Safety First asserts that it is reasonable, and probable in practice, that an individual will seek to reduce as far as is possible the chance of such a catastrophe occurring." (1952, p. 432)

Therefore, investors set a minimum level of wealth to be preserved from a given loss. For example, when investing an amount k, equal to $100,000, investors might want to avoid the disaster loss d, equal to $25,000. They also evaluate investments in terms of wealth multiples: If they invested 100 and the value went up to 300, they achieve a three-times wealth multiple, or ratio. As noted earlier, the investor may achieve that multiple over a period of several years. The way we describe wealth multiples may bring to mind Peter Lynch's famous k-baggers (1989, p. 32 and ss.). Wealth multiples are one type of metric used in entrepreneurship and especially in the evaluation of venture capital portfolios. In this area, multiples are obtained for different vintage funds and differences between allocations and exit values.

At this point, we have presented some essential components of Roy's model: small, individual investors who are overwhelmed and in a hostile environment, trying to ascertain wealth multiples over a possibly extended investing engagement, etc. Yes, the ensemble still represents an innovative approach to risk management in each of its facets – investing in a few assets to be able to use information to manage risk, use liquidity essentially as "working capital," and avoid spreading oneself thin. We return to this essential aspect of Roy's model in the next part, but first, we offer clarification concerning wealth multiples. Keep in mind as well that Roy's components fit the profiles of specialized angel investors and venture capitalists.

The components of Roy's model mentioned in the previous paragraph play an important role in investing and have been employed in other works in the literature to ascertain different areas of investing. For example, in their study on the optimal of investments to hold, Fisher and Lorie (1970) use multiples for the following reasons:

1. It is hard to understand the effects of different rates over long periods.
2. Returns are misinterpreted – e.g., the assumption that one can deduce the wealth ratio from the knowledge of the mean rate of return.
3. One can use prices directly and commissions and other costs.
4. There is no need to assume normality – this observation rhymes well with Roy's considerations regarding possible nonnormality in investing.
5. There is some ambiguity concerning the frequency of returns to be used in mean-variance analyses: it was yearly in Markowitz (1959) but set to monthly later on (as in Fabozzi, Gupta, and Markowitz, 2002).

As Fisher and Lorie (1970) carefully noted, when using wealth ratios, "[i]t is important to remember that initial equal investments were made in each stock included in any portfolio and that there was no subsequent reallocation of resources to preserve the equality of investment. This is not an investment strategy we advocate; again, it was chosen to make certain that the distributions were affected only by the number of stocks in the portfolio" (p. 185).

Roy did not say anything about allocation changes between the initial period and the end period. As Fisher and Lorie (op. cit.) did, assumed the initial allocations do not change. This assumption proved to be critical in both applied investing and when assessing the reliability of alternative models and research assumptions.

Exhibit I, one of Roy's charts, introduces both the minimal preliminary data/variables setting in Roy's model and his vision and strategy. The top of the chart presents information about the four investments we considered. Four common stocks out of the San Francisco Bay Area in California: Cisco Systems (San Jose, California; CSCO), Nvidia Corporation (based in Santa Clara, California; NVDA), Genomic Health (Redwood City, California; GHDX), and Netflix, Inc. (Los Gatos, California; NFLX). We used initial (3/1/2014) and final prices (2/1/2019), from which we calculated wealth multiples for a hypothetical five-year investment engagement (we offer more information about the stocks later). The investor tried to optimize his total wealth (m) for the period of reference by allocating $100,000 (k) to the four securities while avoiding a disastrous loss of $25,000 (d). In our parallel implementation of the mean-variance model, for each common stock, we collected 60 monthly closing prices adjusted for dividends and splits for the same five-year period. Next, we calculated monthly returns in the usual way (natural logarithms price relatives) to obtain our mean-variance/covariance indicator, which we used to optimize that portfolio. At a certain point, we introduced a 4% risk-free rate. Mean-variance employs rates of returns, and Roy (1952) used dollar values. The computations, therefore, use different coordinates but a simple scaling (linear, affine transformation) makes it very easy to compare both approaches while completely preserving the financial/economic meaning of each.

In the chart, Roy attached a probabilistic interpretation to the situation. The expected value of the end-of-period wealth and disaster avoidance depends not only on expected multiples but also on their likelihood, which can be evaluated by attaching probability theory relationships to the variables and numbers employed. The curve represents the mean-variance tradeoffs, and the straight line the tradeoff between different liquidity-risky assets positions. A

100% position in cash avoids disaster but does increase the initial wealth much (he did not use a risk-free rate). But a combination between the risky assets, bought in appropriate proportions, and cash can both maximize initial wealth and avoid disaster. The optimal combination is represented by point P.

Exhibit I. Roy’s (1952) model setting and summary chart.

		CSCO	NVDA	GHDX	NFLX
	Date	Adj Close	Adj Close	Adj Close	Adj Close
1	3/1/2014	19.089045	17.08966	26.34	50.29
60	2/1/2019	51.77	154.1012	75.97	358.1
	Weath multiples	2.712026715	9.017216	2.884207	7.1207

Total allocation	$ 100,000.00	k
Disaster to avoid	$25,000	d

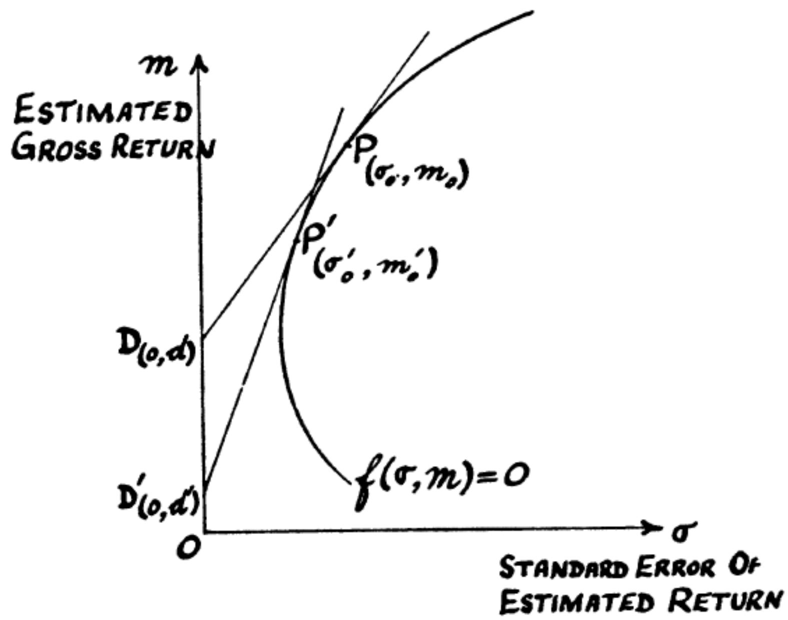


FIGURE 1—The graphical determination of the best σ, m combination.

Chart source: Roy’s (1952, p. 452)

1.2 Probability Theory Component

Roy set the mean-variance relationship and the straight line in probabilistic, symbolic terms, for which he found the tangent to the relevant segment of the conic – a parabola – represented by the curve. His approach could be described as “the hard way, straight up the mountain.”

One strategy to understand his approach is to set up and manipulate a corresponding Lagrangian optimization. An easier way to proceed is to replace multivariate calculus with straightforward linear algebra – always a trustworthy shortcut. Let

$$F(w, V, r) = -1/2\, w'V\, w + w'r \tag{1}$$

represent the portfolio optimization problem, where w is a (k by 1) vector of optimal portfolio weights, V is the variance-covariance (k by k) matrix, and r is a (k by 1) vector of returns. Instead of using (1), we can employ an auxiliary formulation, Z(z, V, r), where z (k by 1) is a vector of auxiliary variables. To solve Z, we compute z =

inv(V) r , then we compute optimal portfolio weight by normalization, wi = zi / sum(z), and we can use the optimal weights to compute portfolio returns, rp = w' r, and portfolio variance, pvar = w'V w. The algebraic expression in (1) implicitly incorporates the investor's (quadratic) utility (Constantinides and Malliaris, 1995), which eliminates the troubles of handling yet another mystifying curve (that of utility).

If we start with a set of n variables, optimizing might reveal that some of the variables have negative weights. These negatively weighted variables are not acceptable in investments and can be avoided by a) employing mathematical programming methods (e.g., Wolfe's quadratic programming algorithm), b) advanced optimization algorithms (e.g., Markowitz's critical line), or c) simply peeling off those variables that have negative weights. The third option has been recognized since Martin (1955), but only Francis and Archer (1971) seem to have revisited Martin's method and used it to enhance teaching and learning about mean-variance optimization (see Tarrazo, 2009, 2014, 2018, in that order).

We can conduct the optimization process in two ways, numerically or analytically (i.e., using letters or symbols). Numeric optimization quickly produces a few numbers of interest, such as values for portfolio returns and risk or the value of the initial function. For example, we can substitute the optimal values for z in Z (z, V, r). That would be Z* = ½ z' inv(V) V inv(V) z + z'V-1 V z, and, since the first order conditions require V z = r, it would simplify to Z * = ½ z'V z. The analytical way is more complicated but it can represent any numbers. Roy worked with the ellipse -1/2 z'V z + z'r – Z = 0, and found the point at which it touches the tangent represented by the straight line in the graphic. Numerically, this is quite manageable and is a typical problem of college mathematics (see for example Kung, 2003). Analytically, however, it is quite involved, gets rather messy, and may represent the battlefield where many brave attempts to understand Roy's model found their end. In a way, Roy's work is an extension of an equally condensed account of what Cramér (1999, originally published in 1946) referred to as "the ellipse of concentration." (An ellipse is a closed curve that, when represented in the mean-variance plane, appears as a parabola, and in the mean-standard deviation plane, as a hyperbola. The three of them are instances of conics).

Roy opted for the hardest approach because of his superb mathematical probability skills. Readers interested in replicating his analysis might find the expositions by Huang and Litzenberger (1988, Chapter 3, p. 59 and ss.), Merton (1972), and Roll (1977) helpful. These works used standard mean-variance analyses. Levy (1973, 1976) does use the correlation version used by Roy (1952), and his work can be read as extensions of Roy's seminal work. However, using algebra is easier and provides some alternative learning.

As we mentioned in the introduction, referencing a straightforward variety of the mean-variance model, such as the one used above, makes unpacking and learning from Roy's model a great deal easier. Mean-variance analysis has tended to use the complicated (semi-definiteness, numerous slack variables, nonnegativity constraints, etc.), presumably in pursuit of the power to analyze hundreds of securities and of theoretical rigor and "robustness," when an easier rout is available. The effect of this "inessential generality," as Steinbach (2001) referred to it, is that one cannot see a thing about what the optimization does or what it means for the investment problem. This also happens when one tries to understand optimizations beyond, say, a dozen stocks. In contrast, we created a straightforward setup that is as mathematically correct as any other, the matrix is very nicely behaved (symmetric, positive definite, unique solutions), and one can see what is going on, for example, with respect to individual effects (ri/stdi, individual returns to standard deviations ratios) and group effects where diversification powers reside (see Tarrazo, 2014).

Using the most straightforward method possible not only makes everything easier, as we demonstrate in our study of correlation-based portfolio optimization presented next and in the rest of the study. It also facilitates drawing implications for investing practice.

1.3 Correlation-Based Optimizations

Exhibit II shows a method for optimizing a portfolio of the common stocks mentioned earlier, using both mean-variance/covariance and mean-correlation approaches. It is worth repeating that we collected 60 monthly closing prices adjusted for dividends and splits for the same five-year (3/1/2014-2/1/2019) period, as recommended in Fabozzi, Gupta, and Markowitz (2002). We then calculated monthly returns using natural logarithms of price relatives and obtained the mean-variance/covariance indicators identified with r, for the vector of average returns, and V, for the 4 by 4 matrix, both computed using monthly data. The top of the exhibit shows individual indicators: returns (ri), variance (vari) and standard deviations (stdi), the very important return-to-risk ratio (ri/stdi), and both annualized returns and standard deviations (ri-y = ri-monthly * 12, stdi-yearly – stdi-monthly * sqrt(12). When looking at those yearly returns, the reader might feel it its better just to go and buy them than to put any more time into reading this study; the sale has been made, the motherlode found! And the reader should cherish that feeling, because that is perhaps the only reason the writer of this study himself put a significant amount of time into this study.

Exhibit II. Mean-variance, and mean-correlation optimizations. Unlevered portfolio analysis.

	ri	vari	stdi	ri/stdi	ri-y	stdi-y
CSCO	0.017095666	0.003375217	0.058096614	0.294263	20.51%	20.13%
NVDA	0.036295615	0.012700794	0.112697801	0.322061	43.55%	39.04%
GHDX	0.01760993	0.011411726	0.106825683	0.164847	21.13%	37.01%
NFLX	0.028787223	0.015815636	0.125760233	0.228906	34.54%	43.56%

V = Covariance				**r**
0.003380088	0.00181589	-9.17412E-05	0.002083447	0.017096
0.00181589	0.012722751	0.000996667	0.005213573	0.036296
-9.17412E-05	0.000996667	0.011416895	0.001876718	0.01761
0.002083447	0.005213573	0.001876718	0.015829448	0.028787

	zi = inv(V) * r	wi = zi/sum(zi)	100K invested
z1	3.695689397	**49.06%**	$49,056.01
z2	2.011940513	**26.71%**	$26,706.19
z3	1.312018572	**17.42%**	$17,415.53
z4	0.513963738	**6.82%**	$6,822.27
	7.533612221	1	$100,000.00

	monthly	yearly
rp	0.023110432	**27.73%**
varp	0.003067643	
stdp	0.055386306	**19.19%**
rp/stdp	0.417258955	

C = Correlation				**ri/stdi**
100.0000%	27.5767%	-1.5591%	28.4037%	0.294263
27.5767%	100.0000%	8.1902%	36.6626%	0.322061
-1.5591%	8.1902%	100.0000%	13.9066%	0.164847
28.4037%	36.6626%	13.9066%	100.0000%	0.228906

	zi = inv(C) * r	zi/stdi	wi = (zi/stdi)/sum(zi/stdi)
z1	0.215331881	3.706444575	**49.06%**
z2	0.227401133	2.017795653	**26.71%**
z3	0.140565165	1.315836801	**17.42%**
z4	0.064824303	0.515459475	**6.82%**
	0.648122482	7.555536505	1

So, how does mean variance analysis help us set the best portfolio? We first computed the auxiliary variables zi by pre-multiplying the inverse of the variance matrix by the return vector. We then compute the normalized weights (wi) by dividing each zi by the sum(zi). Our attention shifted to calculation of the portfolio indicators, which are shown both in monthly and annualized terms for convenience: that is, the portfolio return rp = w’ r = 27.73% and the portfolio standard deviation stdp = sqrt(varp) = sqrt(w’ V w) = 19.19%. The ratio rp/pstd = 0.41 can be interpreted as a return of 41 cents per dollar of risk. The standard deviation calculation used the same units as the average, which made it more understandable, but the variance was the natural realm of the algebraic model. Take, for example, the summation of individual zi's, the unassuming auxiliary variables, which equals the portfolio return-to-variance ratio sum (zi) = 7.5336 = rp/pvar. By linearity, that is also the ratio between the marginal return (ri) -to-marginal variance (vi'= V * w; 7.5336 = ri/vi' for each security and for the portfolio). The marginal conditions also show why a security with negative returns would never be included in an optimal portfolio with positive returns. The sum (zi) would also be the inverse of the Lagrangian variable we could have used to obtain the individual portfolio weights, the standardizing variable, and the return-to-risk slope. A security with a negative return would have had a negative marginal return-to-risk ratio and would have been excluded from the portfolio using, for instance, slack variables. An optimal portfolio would have to have a positive payoff, rp > 0, relative to risk; otherwise, it would not make sense to invest in it. A positive return is a necessary, but not sufficient, condition for the inclusion of that security in an optimal portfolio. The security still needs to increase the return-to-risk ratio of the portfolio.

Thanks to our compact algebraic approach, we can illustrate the mean-correlation set up in one paragraph. We calculated the correlation table, which shows the portfolio's (100%) value in the main diagonal (north-west/south-east) and facilitates the identification of beneficial covariance effects (close-to-zero values, the insurance effect; negative values, the pure diversification effects). On the right-hand side, we used standard-deviation-adjusted returns. After obtaining the auxiliary variable, we divided them by individual standard deviations, and we obtained the optimal portfolio weights, wi = (zi/stdi)/sum(zi/stdi), which are, of course, identical to the mean-variance ones.

In the correlation setup, we experienced visual gains concerning the risk-matrix, but we lost sight of the returns. For cases without data available to compute variance-covariance, the correlation set up becomes attractive because experts could approximate ri, the standard deviation (stdi, expressed in the same units than the mean), and the correlations. The portfolio message is direct and powerful: out of a $100,000 budget, invest $49,056.01in CSCO, $26,706.19 in NVDA, $17,415.53 in GHDX, and $6,822.27 in NFLX.

1.4 Levered-Portfolios and the Value-Based Safety-First Principle

Our next step was to understand levered portfolios – those that can be constructed by mixing the risk-free rate and our common stocks. We also needed to observe the precise manner in which those levered portfolios help manage risk.

Exhibit 3 has two components. The top is a graphic similar to Roy's chart. The curve represents (rp-pstd) portfolios that we can compute using the unlevered (no risk-free rate exists) numbers shown in Exhibit II. Using the no-risk-free-rate numbers of Exhibit II, we calculated the highest return-to-risk possible portfolio (rp = 27.73%, pstd = 19.19%, rp/pstd = 0.4172) – called the tangent portfolio of the unlevered world – from the origin (0,0) point to the (rp=27.33%, stdp = 19.19%) point.

The presence of a risk-free asset changes more things than one might anticipate. First, it allows investors to mix risk-free earning cash by combining a slightly higher return-to-risk portfolio with risk-free holdings that earn a given return (rf = 4%, yearly). If investors allocate the whole $100,000 into the stock portfolio, they would earn 28.84% = rp, and a pstd = 19.99%, for a ratio (pr-rf)/pstd = 0.3588. This is equivalent to point P in Roy's chart, and the tangent would go from the (rp = rf = 4%,0) point to the (rp = 28.84%, stdp 19.99%) point. This new tangent portfolio does not seem that impressive; what is impressive is that we move up and down the straight line, which provides better investment opportunities, such as the following:

1. Put the $100,000 into cash, which means lending money (wr = allocation to risky assets = 0, wf = allocation to the risk-free rate = wf = 100%), and obtain 4% yearly returns, avoiding a disaster loss, in Roy's terms). But, of course, that would yield not equity gains.
2. Put 50% into risky securities and lend 50% as cash (wr = 50%, wf = 50%), which would provide risky returns of 14.42% (0.5 * 28.84, the tangent returns), with a spstd = 9.99%, half the stdp of the tangent portfolio, and earn riskless returns of 2% (4% * 0.5).
3. Allocate 100% to risky securities and earn the 28.84% of the levered tangent portfolio.
4. Or, invest $200,000 in the portfolio, the initial $100,000 and another $100,000 borrowed at 4% cost. This would mean wr = 2, wf = -1, wr + wf = 2 – 1 = 1, rp = 2* 28.84%, and pstd = 39.98% = 2 * 19.99%.

This is impressive indeed. Once we calculated the levered tangent portfolio, it was easy to evaluate the expected returns and risks of different positions and customize the desired/needed returns and risks. Note that allocations to cash were not part of a passive, secondary thought but part of the primary, active investment strategy.

Levered returns and risks can be calculated using the tangent portfolio, but they can also be calculated based on optimal weights, as we did in the lower part of the exhibit. For this method, we first calculated risk-free adjusted returns (ef = ri – rf) and computed our auxiliary variables, z = inv(V) * ef. Each zi was then adjusted by a factor (desired rp - rf)/h, where h = ef' V ef. Every time we changed our desired return, the zi's did not change but the adjusting factor did, which made the sum of portfolio weights adjust correspondingly. That is, a 50% investment in risky assets corresponded with a return of 14.42%, and wr = 0.5 – note that the addition of the optimal weights in the wr = 0.5 (third column from the right) 22.33% + 15.43% + 8.27% + 3.97%) added up to desired 0.5.

Once again, the algebraic approach has allowed us to illustrate something rather involved in the space of a single page.

Let us now compare the mean-variance approach to Roy's model.

To minimize the probability of disaster loss d, Roy needed to maximize the probability of gaining at least the value of d. This led him to maximize the portfolio value net of losses, that is [(m – d) / pvar] using Chebyshev's inequality. Mean-variance analysis expresses the same, using (rp-rf)/pstd. To demonstrate Roy's approach, we used a straightforward version of the aforementioned inequality, which relates the loss to a given value (actual value x minus the expected value E{x} = m-d). This is: P(| x−E{x}|≥ (m-d)) ≤ Var(x)/ (m-d)^2. As usual, Roy chose the hard way.

Next, we computed values, as presented in Exhibit IV. We had the funds to be invested, k = $100,000, and the disaster loss, d = $25,000, which imply a preservation of 75% of the capital. We computed the wealth multiples as explained earlier. The loss/capital ratio (0.25 = $25,000/$100,000) was then subtracted from those multiples.

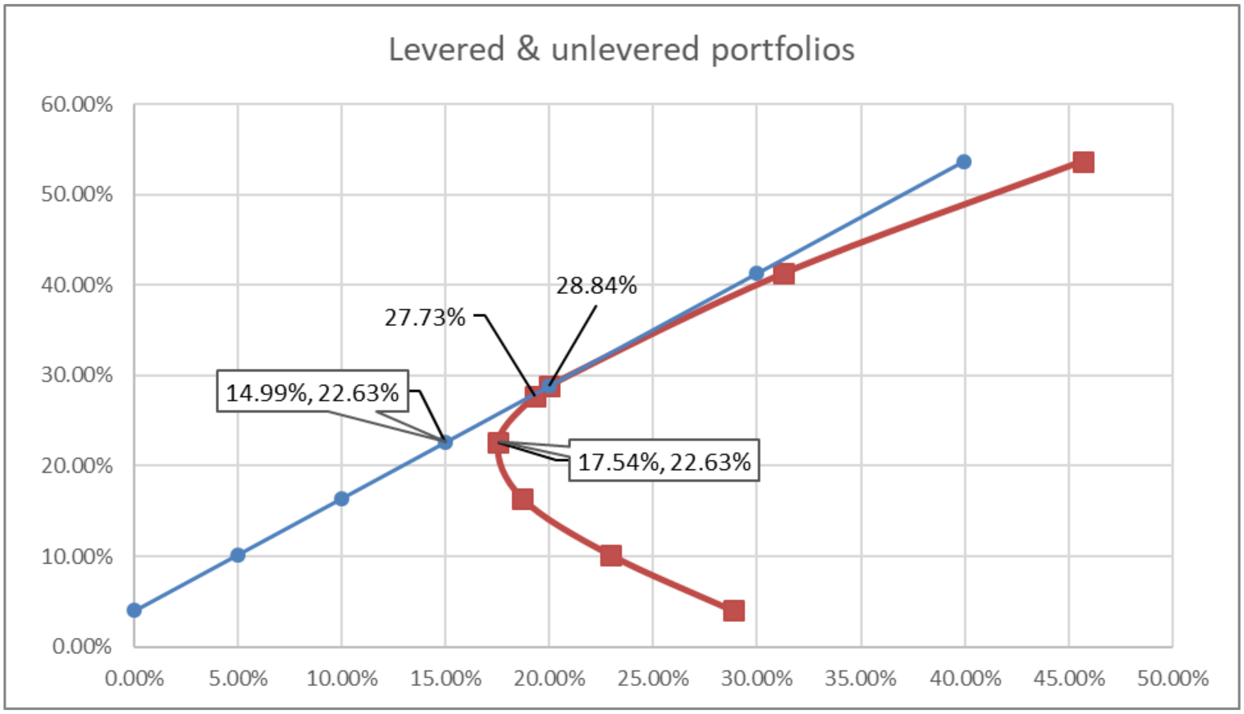
At this point, we faced our first serious challenge: we would have to divide these dollar earnings by their standard deviation, which we cannot calculate in Roy's setting because his model is not supposed to use it. Interestingly, he writes, "If we thought that the estimates had equal proportional reliability, which is perhaps a more plausible assumption in the economic world, we could determine our x's [dollar allocations to each asset] using xi = λ = (pi – d/k) / pi^2" (Roy, 1952, p. 439). Earlier, he noted that, situations with no correlations between the prices of assets would lead to allocations such that xi = λ = (pi – d/k) / αi^2 (equation 8 on the same page). This suggests we could use the square of the wealth multiple as the variance and the wealth multiple itself (pi) as the standard deviation αi. Roy understood that the usual standard method could not be calculated with the usual descriptive statistics. In the next step, we adjusted these dollar-denominated wealth multiples per asset, as shown in the last (rightmost) column.

Our strategy to meet Roy's challenge included key elements of modern finance: 1) "Other things being equal, the volatility of return of a high-priced stock and the volatility of return of an otherwise similar low-priced stock should be the same, a cross-sectional hypothesis," and 2) "The volatility of return of a given stock typically varies inversely with its stock price, … a time-series hypothesis"(both quotes from Rubinstein, 2006, p. 101.)

This left us with a second, equally serious challenge: Roy's model used something it cannot compute –the asset correlation matrix. For our purposes, we "lent" Roy a correlation matrix representing the mean-variance data. In reality, experts would need to provide the wealth multiples, their standard deviations of wealth multiples (αi's), and the correlation matrix. This means all inputs in Roy's model have to be expectations provided by experts.

Applying the same mechanics as we did in the levered, mean-variance case, we arrived at the xi's, which are the dollar allocations: $16,303.20, CSCO; $6,320.30, NVDA; $22,001.02, GHDX; and $6,627.10, NFLX. That means that out of $100,000 investment, $51,251.63 were allocated to risky assets and $48,748.37 to cash. Multiplying the dollar allocations by the wealth multiples produced the terminal value of the investments: $44,214.71, CSCO; $$56,991.55, NVDA; $63,455.49, GHDX; and $47,189.61, NFLX, all for a total ending portfolio value of $271,161.21.

Exhibit III. Mean-variance levered –include risk-free investing-- portfolios.



drp-yearly	53.68%	41.26%	**28.84%**	22.63%	16.42%	10.21%	4.00%
rf-yearly	4.00%	4.00%	**4.00%**	4.00%	4.00%	4.00%	4.00%
drp-m	0.044734	0.034384	**0.024034**	1.89%	1.37%	0.85%	0.33%
rf-m	0.003333	0.003333	**0.003333**	0.33%	0.33%	0.33%	0.33%
h	0.128702	0.128702	**0.128702**	0.128702	0.128702151	0.128702151	0.1287022
(drp - rf)/h	0.321681	0.241261	**0.160841**	0.120631	0.080420338	0.040210169	0
zi = inv(V) ef	wi	wi	**wi**	wi	wi	wi	wi
2.777239788	89.34%	67.00%	**44.67%**	33.50%	22.33%	11.17%	0.00%
1.918930026	61.73%	46.30%	**30.86%**	23.15%	15.43%	7.72%	0.00%
1.028064093	33.07%	24.80%	**16.54%**	12.40%	8.27%	4.13%	0.00%
0.493098838	15.86%	11.90%	**7.93%**	5.95%	3.97%	1.98%	0.00%
wr	**2**	**1.5**	**1**	**0.75**	**0.5**	**0.25**	**0**
wr ri	57.68%	43.26%	**28.84%**	21.63%	14.42%	7.21%	0.00%
wf = (1-wr)	**-1**	**-0.5**	**0**	**0.25**	**0.5**	**0.75**	**1**
wf rf	-4.00%	-2.00%	**0.00%**	1.00%	2.00%	3.00%	4.00%
rp-y	53.68%	41.26%	**28.84%**	22.63%	16.42%	10.21%	4.00%
varp-m	0.013318	0.007491	**0.003329**	0.001873	0.000832372	0.000208093	0
pstd-y	39.98%	29.98%	**19.99%**	14.99%	9.99%	5.00%	0.00%
(drp-rf)/stdp	0.3588	0.3588	**0.3588**	0.3588	0.3588	0.3588	

Exhibit IV. Roy's (1952) model.

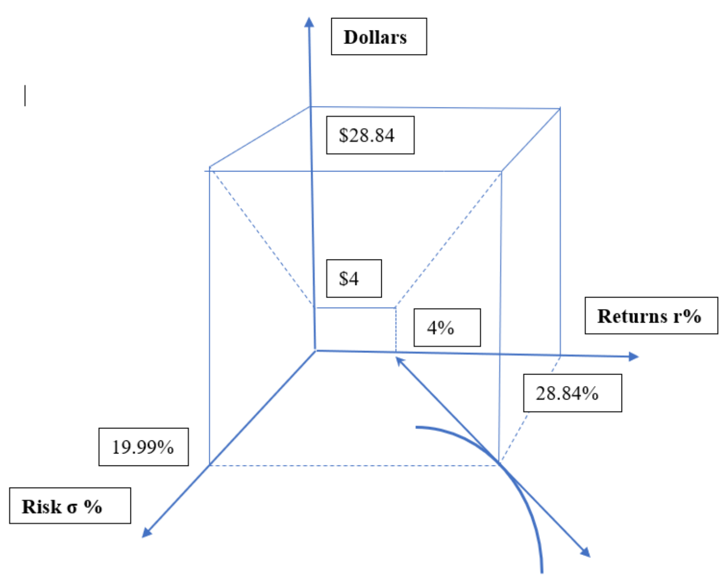
	pi	pi - (d/k)	stdi = ? sqrt(pi^2)=pi	(pi-d/k)/(pi)
CSCO	2.712026715	1.9620267	2.712026715	0.723454052
NVDA	9.017215904	8.2672159	9.017215904	0.916825769
GHDX	2.884206568	2.1342066	2.884206568	0.739963147
NFLX	7.120699918	6.3706999	7.120699918	0.894673275
d	$25,000.00			
k	$100,000.00			
d/k	75%	preservation		

Correlation = ?				**(pi-d/k)/(pi)**
1	0.275767335	-0.015591	0.284037152	0.723454052
0.275767335	1	0.0819015	0.366626353	0.916825769
-0.015590631	0.081901507	1	0.139065821	0.739963147
0.284037152	0.366626353	0.1390658	1	0.894673275

		zi/pi = xi	xi * k	wi	xi * x * pi
z1	0.442147051	**0.163032**	$16,303.20	31.81%	$ 44,214.71
z2	0.569915486	**0.063203**	$6,320.30	12.33%	$ 56,991.55
z3	0.634554936	**0.2200102**	$22,001.02	42.93%	$ 63,455.49
z4	0.471896147	**0.066271**	$6,627.10	12.93%	$ 47,189.61
	2.11851362	0.5125163	$51,251.63	invested	$211,851.36
			$48,748.37	cash	

	t0	t+5	**rcy**
cash	$48,748.37	$59,309.85	4%
securities	$51,251.63	$211,851.36	16.20%
		$271,161.21	**22.08%**

drp-yearly	**25.00%**
rf-yearly	4.00%
drp-m	2.08%
rf-m	0.33%
h	0.128702151
(drp - rf)/h	0.135972864
inv(V) ef	**wi**
2.777239788	**37.76%**
1.918930026	**26.09%**
1.028064093	**13.98%**
0.493098838	**6.70%**
wr	0.8453885
wr ri	24.38%
wf = (1-wr)	0.1546115
wf rf	0.62%
rp-y	25.00%
varp-m	0.002379525
pstd-y	16.90%



The initial $48,748.37 kept in cash became $59,309.85, 4% annualized compound yield – acy = ((final value/initial value)^(1/5)-1). The portfolio of risky assets went from $51,251.63 to $211,851.36, a five-year annualize compound yield of 16.20%. The wealth position went from $100,000 to $271,161.21, a 22.08% annualized compound yield. No disaster loss materialized.

We show two other elements of interest in Exhibit IV. At the bottom left, we show the levered mean-variance with a desired return of exactly 25% for comparison purposes. At the bottom right, we have a graphic illustrating the integration of return and risk percentages with the dollar dimension (the vertical line) using the levered tangent portfolio.

A few observations:

1. Both models meet the minimum required amount, while maximizing remaining wealth.

Roy's model's total wealth acy = 22.08 is similar to the mean-variance model's 24.38% expected returns on risky assets.

2. Roy's model keeps a higher proportion of cash ($48,748,37) than does the mean-variance model ($15,461.15), which seems excessive. This amount might be needed to cover for the very worst-case scenario concerning assets whose values could go into negative territory. The maximum loss from investing in publicly owned common stocks is zero. In Roy's general concept of investment assets, subtracting two standard deviations from the expected value of the portfolio gives a possible 9.41% loss, 24.38% - (2*16.90) = - 9.41%, at a 25% chance of happening. The large cash reserve might also have been caused by using wealth multiples themselves (pi) as proxies for their standard deviation.
3. Roy's model does have some intriguing and meritorious internal consistency.
4. The wealth multiple adjustments remind us of James-Stein shrinkage estimators and might be responsible for the reversal of weights compared to mean-variance analysis: the 64% allocated to CSCO and NVDA is replaced by the 55% GHDX and NFLX.

2. Evaluation, Potential Modern Applications, and Further Research

When we initiated our revision of Roy's model, we expected the effort to provide guidance for enhancing practical investing especially when using portfolios with a small number of securities. We also expected the effort would help us identify research contributions that would further that goal. Our expectations were based on the following items and issues touched upon by Roy's approach, with some incidence in the mean-variance literature.

2.1 Evaluation

Different natures and roles of liquidity. Liquidity could be seen as one thread that guides major developments in economics and finance – see Tobin's (1958) application of mean-variance analysis to cash-holdings to explain Keynesian money demand and how liquidity as an investable asset becomes a stepping stone in the Capital Asset Pricing Model. Yet, the lack of an appropriate conceptualization for household/investor liquidity has hampered progress in portfolio theory, for which a static, one-period, "cash as residual" approach is not practical. As Warschauer and Guerin (1987) put it, "Many elements of the personal financial planning process lack a theoretical basis and so are troubling to both academics and committed professionals in the field. One of the most perplexing questions facing scholars and practitioners alike relates also to the matter of personal liquidity. What is the optimal amount of liquid assets to be held by an individual? This question must be answered in every financial plan. The answer however, is not simple. Consequently, far too often this important subject is dismissed during the planning process through the expedient of applying 'rules of thumb' that have little, if any, basis in theory" (p. 355).

Cash flows and risk. Return-based approaches ignore period-by-period cash flow needs, bypassing the client-specific customization required by small, household investors. The consequence is a misrepresentation of risk for this class of investors because, as noted by Smidt (1978), "(T)he risk of an asset or of a portfolio cannot be determined without knowing something about the characteristics of the investor" (p. 18). Moreover, risk must be defined in terms of consequences of not having expected incomes, as was eloquently and effectively noted by Jeffrey (1984), "The problem with equating portfolio risk solely to the volatility of portfolio returns is simply that the proposition says nothing about what is being risked as a result of the volatility. … Volatility per se, be it related to weather, portfolio returns, or the timing of one's morning newspaper delivery, is simply a benign statistical probability factor that tells us nothing about risk until coupled with a consequence. … The determining question in structuring a portfolio is the consequence of a loss; this is far more important than the chance of a loss. … [the following is capitalized in the original] RISK IS THE PROBABILITY OF NOT HAVING SUFFICIENT CASH WITH WHICH TO BUY SOMETHING IMPORTANT" (p. 34).

Wealth multiples, asset allocation, and the time horizon. "It is quite interesting that the mean-variance approach has received comparatively little attention in the context of long-term investment planning" (Steinbach, 2001, p. 32). However, once cash becomes an investing option, investors can rely on hedging and wait for their portfolio of risky investments to grow. Holding cash, which eliminates risk (100% liquid is 100% without risk), is far more powerful means of hedging than is diversification, which can only partially diminish risk. Time brings a resolution of uncertainty, "When the period is sufficiently long for both yields over the period and prices at the end of the period to be of importance, then the p's (wealth multiples) become our best estimates of the yields and the prices of the various assets and the α's are again the appropriate standard errors" (Roy, 1952, p. 440). Finally, the investor's portfolio may include a variety of assets (real estate, education, fixed income, entitlements and contracting such as health insurance), the consideration of which is essential for the overall management of the investor's risk. The understanding of risk management as simply a matter of holding many stocks is incorrect. Levy (1996) used a correlation set up to examine how different types of securities might play out in different time horizons. Very interestingly, he shows that a mean-variance model might favor positions with simply cash and the most conservative asset in the portfolio. Goetzmann and Edwards (1994) find that, "Four investors with multiple-year investment horizon, summary statistics based upon short-term (annual) returns can be grossly misleading for making asset allocation decisions. In particular, the autocorrelation structure of asset class returns strongly influences both the variance and the interclass correlation of assets, and, as a consequence, changes the composition of the efficient frontier" (p. 76).

Hardship in working on the curved segment instead of simply riding the straight levered-portfolio line. Working on the theoretical probability ellipse is not only difficult but also precludes using cash, which is essential in the risk component of the model (see Russel and Smith, 1966). Something similar happens when trying to encapsulate the investor's situation and decisions with a utility curve.

How Portfolio optimization works. Throughout the optimization process, we observed a term that repeats itself over and over. It is the most direct and powerful expression of what we look for and how we find it: the return-to-risk ratio. We could express the risk as the variance or the standard deviation. Everything boils down to it. In the unlevered case, risk is expressed at the level of the portfolio as rp/pstd and at the level of individual securities as ri/stdi. In the levered case, risk for the portfolio is expressed as (rp- rf)/pstd and for individual securities as (ri – rf)/stdi. In Roys's model, using dollar amounts, risk is expressed as (m – d)/σ or, for each asset, as (pi – d/k)/ αi. Roy also noted that in the case of no correlation, optimal dollar allocations are qual to xi = λ (pi – dk)/ αi^2. For the unlevered, zero-covariances case, wi = λ (ri/vari), with lambda being, λ = 1/ sum(ri/vari), Mao (1970). For the levered case, we would just replace ri with (ri-rf). This means that the return-to-risk ratio is controls the optimization. We could easily show that the ranking of optimal portfolio weight total or partially replicates the ranking of securities by their individual return-to-risk ratio.

Exhibit V shows portfolio optimizations for eight portfolios (we used Portfolio #4 throughout our study). We preselected 10 securities, mostly from the San Francisco Bay area (because that is where money is), and obtained the usual mean-variance optimal portfolio weights. In the each of the portfolios, positively weighted securities have been boxed in. First, note that optimal weights occur practically every time in the upper space determined by the ri/stdi rankings. Second, in some cases, the ri/pstdi ranking is exactly replicated by the ranking of optimal weights. Third, even when it is not the exception, the straight lines (perfect replication of ranking) carry the highest weights. Fourth, diversification effects are indicated by the upward or downward sloping curves. An upward slope represents a reward (a promotion) for a security that makes the heavy weight look better; a downward slope represents a penalty (a demotion) for a security with excessive positive covariance burden. In other words, portfolio optimizations set the inclusion box using individual return-to-risk rankings. Diversification effects come next, upgrading or downgrading the security accordingly.

It is important to note that the rankings and diversifications can only be observed in portfolios with a small number of securities. As the numbers of securities is increased (even beyond 30, the size of the Dow-Jones), estimations errors due to numerical problems (ill definition, multicollinearity, and under- and overdetermination) are likely to make matters unclear. The outcome of inserting 100, 200, 400, or even more securities into an optimizer using brute force is unclear. What seems likely, and can be read between the lines in the literature, is that the optimizer responds by severely cutting down the number of securities, which probably displeases the user who then thinks portfolio analysis is unreliable.

Exhibit V. Diversification and covariance effects.

P1 rp/stdp 0.448666

	ri/stdi		wi*
ADBE	0.395711	ADBE	38.69%
INTU	0.357916	INTU	33.94%
NVDA	0.322061	NVDA	13.86%
INTC	0.252938	INTC	7.96%
APPL	0.233182	FB	5.55%
CRM	0.229921	CRM	0
FB	0.215065	SAP	0
GOOG	0.184156	APPL	0
SAP	0.120186	GOOG	0
CVX	0.068401	CVX	0

P2 rp/stdp 0.345639

	ri/stdi		wi*
CSCO	0.294263	CSCO	38.88%
INTC	0.252938	INTC	31.53%
CRM	0.229921	GOOG	14.98%
GOOG	0.184156	CRM	14.61%
SCHW	0.131283	CVX	0
HPQ	0.130251	GPS	0
WSM	0.130251	BMRN	0
CVX	0.068401	HPQ	0
BMRN	0.022418	WSM	0
GPS	-0.0628	SCHW	0

P3 rp/stdp 0.337578

	ri/stdi		wi*
INTC	0.252938	INTC	46.61%
CRM	0.229921	FB	29.13%
FB	0.215065	CRM	24.26%
GOOG	0.184156	CVX	0
ORCL	0.120788	GOOG	0
CVX	0.068401	TWTR	0
WFC	0.065652	ORCL	0
GPS	-0.0628	GPS	0
TWTR	-0.0679	WFC	0
YELP	-0.1022	YELP	0

P4 rp/stdp 0.417866

	ri/stdi		wi*
NVDA	0.322061	CSCO	49.06%
CSCO	0.294263	NVDA	26.71%
NFLX	0.228906	GHDX	17.42%
GHDX	0.164847	NFLX	6.82%
SSD	0.129527	EXEL	0
EBAY	0.090091	GILD	0
EXEL	0.082611	BEN	0
WFC	0.065652	WFC	0
GILD	-0.0314	EBAY	0
BEN	-0.10703	SSD	0

P5 rp/stdp 0.575406

	ri/stdi		wi*
WCN	0.482923	WCN	51.05%
CSCO	0.294263	CWT	21.20%
CWT	0.259956	CSCO	14.09%
AAPL	0.233182	NFLX	9.44%
CRM	0.229921	AAPL	3.54%
NFLX	0.228906	TSLA	0.68%
EBAY	0.090091	YELP	0
CVX	0.068401	CVX	0
TSLA	0.041844	CRM	0
YELP	-0.1022	EBAY	0

P6 rp/stdp 0.414188

	ri/stdi		wi*
NVDA	0.322061	LMT	27.38%
CSCO	0.294263	CSCO	21.94%
LMT	0.262436	INTC	21.56%
INTC	0.252938	NVDA	14.94%
AAPL	0.233182	ABM	7.28%
CRM	0.229921	TSLA	5.10%
ORCL	0.120788	AAPL	1.80%
ABM	0.103957	ORCL	0
CVX	0.068401	CRM	0
TSLA	0.041844	CVX	0

P7 rp/stdp 0.481861

	ri/stdi		wi*
ADBE	0.395711	ADBE	36.18%
V	0.373323	ROST	24.20%
INTU	0.357916	INTU	23.20%
ROST	0.278786	V	8.13%
AAPL	0.233182	ISRG	4.89%
CRM	0.229921	NFLX	2.05%
NFLX	0.228906	EA	1.35%
EA	0.228506	CRM	0
AMD	0.180101	AMD	0
ISRG	0.100269	AAPL	0

P8 rp/stdp 0.506429

	ri/stdi		wi*
ADBE	0.395711	ADBE	44.58%
DLR	0.27356	CLX	26.57%
CLX	0.241593	DLR	23.19%
GHDX	0.164847	ISRG	2.69%
ATVI	0.156425	BLUE	1.52%
IRBT	0.142902	GHDX	1.46%
RHI	0.139849	RHI	0
BLUE	0.138151	IRBT	0
ISRG	0.100269	ATVI	0
CVX	0.068401	CVX	0

Roy (1952) and early applications of mean-variance analysis favored few investments and applied a first layer of risk reduction using security and company analysis. Tarrazo (2009) shows that the ri/stdi ratio can be used heuristically to build a portfolio sequentially, by starting with the security with the highest ratio and trying out the others with positive return one at a time.

2.2 Modern Applications

Proper revisiting of Roy's (1952) model required an integration with the mean-variance counterpart, not a comparison. Despite the unusually polished final look of Roy's contribution, the mean-variance model underwent both backward and forward powering ups like no other theory in modern finance. For example, it quickly absorbed previous work on utility theory and optimization in addition to mathematical programming, game theory, and input-output analysis. It then benefitted from and helped itself to develop monetary theory, as in Tobin (1958), and capital markets models, all of which used the levered portfolio in search of a very special, but often elusive, tangent portfolio referred to as the market portfolio. Still, the previous section has noted other elements may be further enhanced. This section suggests that enhancement efforts may take household equity investing to level.

Developing a modern theory on direct equity investing by households. In this matter, we do not seem to be in a different situation from that of Roy and Markowitz when they wrote their contributions. It seems we have an implicit understanding (or unwritten supreme directive) that direct equity investing (buying common stocks) is reserved for institutional investors, and individuals have to content themselves with indirect investing (i.e., mutual funds). The retirement setup reinforces that idea, and individuals are led to feed on a straight "diet" of mutual funds though plan "fiduciaries" – a situation in which the "portfolio diversification" mantra often sounds like the "eat your vegetables" mantra," as Riepe (2002) put it.

In some academic quarters, small portfolios are still treated essentially as hobbies, and the efforts of those individuals who invest in common stocks do not seem to be taken seriously or appreciated, as implied in the following text: "The basic premise of this paper is that a substantial number of investors forego a holistic portfolio optimization approach along the lines advocated by Markowitz (1952, 1959), and rather, select stocks sequentially. These are people who, exhibiting narrow framing, evaluate one stock at a time, or perhaps compare the relative merits of one stock versus another." In other words, the authors seem to have already decided that investors exhibit narrow framing – whatever that may actually be – in a pejorative sense and that no matter what they do, it is already wrong, even though no guidance seems available other than a strategy described as "holistic," whatever that may mean. In the same quarters, criticisms of the admittedly limited diversification in small portfolios ignore the way using information effectively may help control risk and see properly following and integrating analysis of a few stocks as impossible. They also ignore whether those investors' asset allocations are suitable for the risk inherent in their equity investing. In addition, many studies do not seem to refer to any actual investing at all but resemble flights of fancy where one thing (e.g., assuming the investor selects stocks at random [although selecting stocks at random is a contradiction in terms]) leads to another, using equal weights, to end up recommending the use of securities galore. The unrealistic assumptions may originate in technical items divorced from financial meaning. For example, see the following quotations: (a) "Consider, for simplicity, the case in which the standard deviation, of each of the n stocks is identical; here all correlations, p, between pairs of stocks are identical; and where the weight of each of the n stocks is the same. In this case …." (b) "Now, assume that all the stocks have the same expected return…." (Author note: Names omitted on purpose on the last two quotations.)

Fortunately, there is a tradition of authors who have tried to help those brave individuals who invest in stocks. These authors include practitioners – "stalking the tenbagger" Lynch (1989), Loeb (1965), and Chhabra (2004), who stated there exists "a minority of the general population but a significant majority of the affluent, who have become wealthy over the years by defying market diversification. Ironically, many of them would not have become so wealthy had they followed a conventional diversified approach," (2005, p. 8).

Such a tradition of helping individual small investors also includes a few academic researchers. In its modern (quant) form, the field was initiated by many accomplished scholars such Markowitz (1952, 1959), Tobin (1958), and Roy (1952), and it pulsates with many contributions by William Sharpe. It reached a high-water mark with Jacob (1974) and the entire research program of Elton and Gruber: see Elton, Gruber, Brown, and Goetzman (2010). Part of the research program by Haim Levy bridges the historical span from Roy's original article to today. In addition, it employs (or provides) the building blocks of modern finance that may help build such a modern theory of household equity investing; for example, Levy (2004, 1996, 1979, 1976, and 1973). Levy (1976) may be the portfolio model most suitable for small, individual household investors. By the way, a very useful presentation of the levered mean-variance model can be found in Levy and Sarnat (1984), especially Chapter 9: "Tracing the Efficient Frontier."

We should keep in mind that mean-variance analysis, as in Markowitz (1959), also seems to have been written for a variety of investors, including household ones. In fact, his "Illustrative Portfolio Analysis" is still unsurpassed in the way portfolio theory can help those investors. In that chapter, Markowitz applies his model to nine securities, using 24 yearly rates of returns. While some may think this is probably an outdated detail from early applications of the theory, the tightness between the number of securities and the number of observations addresses some of the problems of convergence (under- and overdetermination) estimation errors that plague portfolio optimizations to this day; see Narowcky (2000) and López de Prado (2016). That early example also includes security preselection efforts, excludes non-negative weights (and returns), and keeps in mind customization: "No single type of analysis is right for all purposes. The choice of analysis depends on the nature and goals of the investor" (Markowitz, 1959, p. 33).

In fact, one may think that a modern theory of equity investing by households may not be far away. We are already able to include the key aspects required in a reasonably complete way: using wealth and allocations in dollar values, which creates room for costs and management considerations; integrating asset planning, including wealth multiples and different aspects of liquidity; and deploying risk management at various levels (household situation, information to assess the investment objects, mean-variance diversification, and hedging through the cash position). In addition, what we thought was an unsurmountable problem in Roy's model – lack of repetition-based data to compute descriptive statistics – opens the door to expectation-built inputs.

A modern theory of equity investing by households would be very good given some developments in our economy. It is time to mention, albeit very briefly, the varieties of equity investing originated at the workplace level as well as crowd-equity, angel, and venture capital investing.

Households, equity, and the startup economy. Equity investing goes well beyond investing in a few common stocks. It can include small and family businesses, franchises, and various forms of employee profit sharing, as well as access to shares through employee stock option plans (ESOPs) and KSOPs – a qualified retirement plan that combines an ESOP with a 401(k), cooperatives, and private equity participation. Investors have many ways to access equity. Access to equity may be the best way to enhance equity and equality in our society. Access to equity represents direct access to the "available for common" cash flow that provides the success of firms to their owners, although it technically distributes as a residual. Imagine what having enhanced access to equity participation in the post-World War II United States could have meant for rank and file employees – we are talking about direct access to the economic results from major endeavors such as the reconstruction of the world after the war, post-war growth, and the revolutions in agriculture, computers and the internet, and biotechnology. Some of that equity would have been non-publicly traded, which would have diminished it attractiveness to employees. However, during the last few years, the internet has created platforms to enhance the liquidity (trading) of privately owned shares in startups.

Furthermore, in the United States, the Jumpstart Our Business Startups Act, or JOBS Act, which passed with bipartisan support, was signed into law by President Barack Obama on April 5, 2012. Its objective was to encourage and facilitate funding of small businesses and startups. One way to do so was to ease regulations, for which it created two types of investors. The first is the "accredited" investor, who has more than $200,000 in income in each of the last 2 years ($300,000 if married) or a net worth exceeding $1 million, not counting the primary residence. The other type is the "non-accredited" investor who, if they earn less than $100,000, can choose between the greatest amount of either 5% of income or 2% of net worth. If the net worth and/or income are more than $107,000, they can choose the lesser amount of 10% on either one, to a maximum of $107,000 (see FINRA entry in the references section). For non-accredited investors, the JOBS Act set the provisions for equity crowdfunding, which lets small investors invest as little as $1,000 in startups through FINRA-regulated internet portals (see FINRA entry in the references section).

Angel investing. In 2009, Shane published a thorough study showing that angel investors were not magical creatures but simply unassuming and efficient business people who help many types of businesses. They have less money and "glamour" than non-business people may have initially believed. Five years later, the picture of angel investing was completely different because the passage of the JOBS Act in 2012 led to a new type of angel investor who uses angel groups and angel investing platforms; see Rose (2014), who is founder and chairman emeritus of one major group (New York Angels), as well as the founder and CEO of gust.com, a most innovative and forward-looking business entity the likes of which has never been seen before. A directory of angel groups is available at the website of the Angel Capital Association (see the association's entry in the references).

These accredited investors are not the dummies individual equity investors are taken to be in the financial research mentioned earlier. They are well educated, having industry backgrounds (often with entrepreneurial backgrounds). They can cover the legal, financial planning and asset management aspects and have enough funds to face the challenge of making money by investing in startups in the best possible conditions. Echoing Chhabra's comment above, Rose (2014) states, "All startups are risky, and it is difficult—if not impossible—to trade off the risk factor against anything else.… My approach, therefore, is the opposite: I have decided that since angel investing already represents a small, specific part of my otherwise-diversified investment portfolio, it does not make sense to diversify further within my angel investments. I approach angel investing with a particular investing thesis, find companies that fit that thesis, and distribute my risk by investing in a number of them," Rose (p. 137). This book also contains a chapter entitled "The Portfolio Theory of Angel Investing: Why Every Angel Needs to Invest in at Least 20 Companies," which reads as an integration of mean-variance/safety risk approaches.

Venture capital. Our exposure to entrepreneurial finance greatly influenced our visualization and understanding of Roy's model, especially concerning the wealth multiples (referred to as return multiples on invested capital), initial allocations, and the specialization in industries, sectors, areas, etc. Of course, many differences exist between investing in common stocks and in startups. For example, in a venture setting, some ventures are liquidated and that provides a lower bound for the worst possible case. However, liquidity is strategically used both to propel survivors further and to optimize opportunity cash flows. Not all the money of the fund would be allocated at the beginning of the investing horizon (usually 10 years): "Imagine that a VC invests $1 million in the first round of your company. At the time of making the investment, the VC reserves a theoretical future amount of the fund to invest in follow-up rounds" (Feld and Mendelson, 2012, p. 124). After some period and evaluation of progress, some companies receive a "double-down" and others receive less than the initial investment or nothing.

The VC pros (amateur VCs do not survive) may already operate in something very analogous to Roy's setup, almost at each step of their business: in choosing a small number of startups preselected according to most stringent financing criteria, in establishing a given level of initial funds, management of reserves, and dollar allocations over time, and in determining carried interest and valuation of equity (ownership). All of this is done with agreements and expectations – "rough numbers" such as multiples, a variety of investing horizons for each startup, etc. With respect to agreements, Lerner et al. (2012) state, "One of the chief concerns is the amount of the fund invested in a single company. For both VC and buyout funds, the agreement generally stipulates a 'concentration limit,' or the percentage of the fund (usually based on committed capital) that can be invested in a single company. This limit guards against the possibility that the GPs [general partners] might focus excessive amounts of time and additional capital on a company in which it had already invested a large amount. Another related concern involves the possibility that GPs might try to offset a large investment in a troubled company by engaging in high-risk strategies with the balance of the fund" (p. 35). Other restrictions limit the use of debt, investments in invested companies from other vintages, and the reinvestment of profits. However, expectations are critical in selecting winners from suitable areas/industries (e.g., software, hardware, medical equipment, human/agricultural or animal biotechnology, or entertainment). Moreover, remember, "When the period is sufficiently long … wealth multiples become our best estimate," and be mindful that the "limitation" of Roy's model (not making use of historical statistics) can be overcome using expert inputs.

Nawrocky (2000) observes mean-variance analysis was developed by academics to explain the historical record, while practitioners would be happier with approximate – but forward-looking – models that are better at forecasting. Our interpretation of Roy's model adds weight to this observation.

In any case, it seems one "hits the pot" in considering venture capital as a potentially suitable area for deploying Roy's model.

3. Concluding Comments

This study originated from a paradox: Roy (1952) being regarded as one of the cornerstones of portfolio theory but nobody having reported any direct implementations of his model. That is what we have done, and it is the first contribution of our study.

We first clarified Roy's objectives, the variables he used, and the structure of his model, and we implemented it numerically.

The standard or canonical mean-variance model established by Markowitz is well known. It is a single period model that uses statistical theory to derive an optimal investment position using a vector of average returns and the variance-covariance matrix returns. Roy begins with the same theoretical, statistics-based conic to formulate a model that would optimize investing revenues for a given level of risk over a period. Roy makes a few departures from the canonical mean-variance problem: 1) wealth multiples for a given time range, instead of single period average returns; 2) dollar-denominated indicators instead of percentages; and 3) outlining a more general way to manage risk whereby pursuing the most attractive expected wealth ratios is tempered by keeping a substantial calculated cash reserve.

Unfortunately, while each of the departures makes a key contribution to potentially improving investing techniques, a steep price is also paid: first, in being able to visualize the whole model and, second, in trying to piece back the impressive risk fabric represented by the variance-covariance structure. To implement Roy's model, one has to break away the wealth multiples (multiperiod) from the usual frequency-based sampling approach to understand a position's risk. In sum, Roy (1952) appears to be a brilliant vision whose components cannot fit the traditional probability theory suit of armor initially designed for them.

However, we are not done yet. It is a fact that certain investors (e.g., venture capitalists) face phenomenal odds in their investments, and they do employ wealth multiples expecting uneven results, whose phenomenal risks they also manage with qualitative as well as some quantitative information and with massive amounts of liquidity, strategically deployed in a multiperiod setting. Of course, this brings to mind Roy's approach, which at some point seemed to indicate that all inputs in Roy's model had to be expectations provided by experts.

While trying to compare Markowitz's (1952) mean-variance analysis to Roy's (1952) model, we actually introduced a hybrid model that admits both qualitative and quantitative components, thus offering the potential to integrate both approaches and perhaps produce some practically useful synergies.

References

ACA. Angel Capital Association, angel groups: https://www.angelcapitalassociation.org/directory/#1

Chhavra, A. (2005). Beyond Markowitz: A Comprehensive Wealth Allocation Framework for Individual Investors. *The Journal of Wealth Management, 7*(4), 8-34. https://doi.org/10.3905/jwm.2005.470606

Constantinides, G. M., & Malliaris, A.G. (1995). Portfolio Theory." Chapter 1, pages 1-30, in Robert Jarrow et al., Handbook in OR & MS, 9. Elsevier Science, New York, 1995. https://doi.org/10.1016/s0927-0507(05)80045-3

Cramér , H. (1999). Mathematical Methods of Statistics. (PMS-9). Princeton University Press, New Jersey. (Originally published in 1946.) https://doi.org/10.1515/9781400883868

Elton, E., Gruber, M., Brown, S., & Goetzmann, W. (2012). *Modern Portfolio Theory and Investment Analysis*. 8th Ed., John Wiley & Sons, Hoboken, New Jersey.

Fabozzi, F., Gupta, F., & Markowitz, H. (2002). The Legacy of Modern Portfolio Theory. *The Journal of Investing, 11*(3), 7-22. https://doi.org/10.3905/joi.2002.319510

Feld, B., & Mendelson, J. (2012). *Venture Deals*. Second Ed., Wiley, New Jersey. https://doi.org/10.1002/9781119204152

Fisher, L., & Lorie, J. (1970). Some Studies of Variability of Returns on Investments in Common Stocks. *The Journal of Business, 43*(2), 99-134. https://doi.org/10.1086/295259

FINRA. US Financial Industry Regulatory Agency.

Regulations: https://www.finra.org/investors/alerts/crowdfunding-and-jobs-act-what-investors-should-know

Crowdfunding portals: https://www.crowdfundinsider.com/2016/11/92810-now-21-finra-approved-reg-cf-crowdfunding-portals/

Francis, J. & Archer, S. (1971). *Portfolio Analysis*. Prentice Hall, Englewood Cliffs, New Jersey.

Goetzmann, W. & Edwards, F. (1994). Short-Horizon Inputs and Long-Horizon Portfolio Choice. *The Journal of Portfolio Management, Summer 1994, 20*(4), 76-81. https://doi.org/10.3905/jpm.1994.409486

Huang, C. F. , & Litzenberger, R. (1988). Foundations for Financial Economics, North Holland, Netherlands.

Jeffrey, R. (1984). A New Paradigm for Portfolio Risk. *The Journal of Portfolio Management, 11*(1) (Fall), 33-40.

https://doi.org/10.3905/jpm.1984.408990

Kung, S. (2003). Finding the Tangent to a Conic Section Without Calculus. *The College Mathematics Journal, 34*(5), p 394. https://doi.org/10.2307/3595824

Kwan, C. (2007). A Simple Spreadsheet-Based Exposition of the Markowitz Critical Line Method for Portfolio Selection. *Spreadsheets in Education, 2*(3), Article 2. 253-280. Available at: http://epublications.bond.edu.au/ejsie/vol2/iss3/2

Lassance, N. (2022). Reconciling Mean-Variance Portfolio Theory with Non-Gaussian Returns. *European Journal of Operational Research, 297*, 729-740. https://doi.org/10.2139/ssrn.3664049

Lerner J., Leamon A., & Hardymon, F. (2012). *Venture Capital, Private Equity, and the Financing of Entrepreneurship*. 1st Edition. John Wiley & Sons, Hoboken, New Jersey.

Levy, H., & Levy, M. (2004). Prospect Theory and Mean-Variance Analysis. *The Review of Financial Studies, 17*(4), 1015-1041. https://doi.org/10.1093/rfs/hhg062

Levy, H. (1996). Investment Diversification and Investment Specialization and the Assumed Holding Period. *Applied Mathematical Finance, 3*(2), 117-134. https://doi.org/10.1080/13504869600000006

Levy, H. (1979). Does Diversification Always Pay? *TIMS Studies in Management Science, 11*, 63-71.

Levy, H. (1976). Equilibrium in an Imperfect Market: A Constraint on the Number of Securities in the Portfolio. *The American Economic Review, 68*(4), 643-658.

Levy, H. (1973). The Demand for Assets Under Conditions of Risk. *The Journal of Finance, 28*(1), 79-96. https://doi.org/10.1111/j.1540-6261.1973.tb01347.x

Levy, H., & Sarnat, M. (1984). *Portfolio and Investment Selection: Theory and Practice*. Prentice Hall International. Hoboken, New Jersey.

Lo, Andrew, & Foerster, S. (2021). In Pursuit of the Perfect Portfolio: The Stories, Voices, and Key Insights of the Pioneers Who Shaped the Way We Invest. Princeton University Press, 2021. https://doi.org/10.1515/9780691222684

Loeb, G. (1965). *The Battle for Investment Survival. First published in 1935, updated in 1957 and 1965*. Wilder Publications, Blacksburgh, Virginia.

López de Prado, M. (2016). Building Diversified Portfolios that Outperform Out of Sample. *The Journal of Portfolio Management Summer, 42*(4) 59-69. https://doi.org/10.3905/jpm.2016.42.4.059

Lynch, P., & Rothchild, J. (2000). *One Up On Wall Street*. Revised Edition. Simon & Schuster, New York.

Mao, J. (1970). Essentials of Portfolio Diversification Strategy. *The Journal of Finance, 25*(5), 109-1121. https://doi.org/10.1111/j.1540-6261.1970.tb00871.x

Markowitz, M. (1999). The Early History of Portfolio Theory: 1600-1960. *Financial Analysts Journal, 55*(4), (Jul. - Aug., 1999), pp. 5-16. https://doi.org/10.2469/faj.v55.n4.2281

Markowitz, H. (1987). *Mean-Variance Analysis in Portfolio Choice and Capital Markets*. Basil Blackwell, Hoboken, New Jersey.

Markowitz, H. (1959). *Portfolio Selection: Efficient Diversification of Investment*. John Wiley & Sons, New York.

Markowitz, H. (1942). Portfolio Selection. *The Journal of Finance, 7*(1), 77-91. https://doi.org/10.2307/2975974

Martin, A. (1955). Mathematical Programming of Portfolio Selections. *Management Science, 1*(2), 152-166. https://doi.org/10.1287/mnsc.1.2.152

Merton, R.C. (1972). An analytic derivation of the efficient portfolio frontier. *Journal of Financial and Quantitative Analysis, 7*, 1851-1872. https://doi.org/10.2307/2329621

Milevsky, M. (2002/2003). Space–time diversification: which dimension is better? *The Journal of Risk, 5*(2), 45-71. https://doi.org/10.21314/JOR.2003.075

Nawrocky, D. (2000). Portfolio Optimization, Heuristics, and the 'Butterfly Effect'. *Journal of Financial Planning*, February 2000, 68-78.

Riepe, M. (2002). Diversification, Vegetables and Bill Gates. *Journal of Financial Planning*, (may 2002), 3638.

Roll, R.(1977). A Critique of the Asset Pricing Theory Tests. Part I: On Past and Potential Testability of the Theory. *Journal of Financial Economics, 4*(2), 129-176. https://doi.org/10.1016/0304-405x(77)90009-5

Rose. D. (2014). *Angel Investing*. John Wiley & Sons, Hoboken, New Jersey.

Roy, A. D. (1952). Safety First and the Holding of Assets. *Econometrica, 20*(3), 431-449. https://doi.org/10.2307/1907413

Rubinstein, M. (2006). *A History of the Theory of Investments: My Annotated Bibliography*. Wiley, Hoboken, New Jersey, 2006. https://doi.org/10.1002/9781119201809

Rubinstein, M. (2006). Markowitz's 'Portfolio Selection': A Fifty-Year Retrospective. *The Journal of Finance, LVII*(3), 1041-1045. https://doi.org/10.1111/1540-6261.00453

Russell, W., & Smith, P. (1966). A Comment on Baumol (E, L) Efficient Portfolios. *Management Science, 12*(7), 619-621. https://doi.org/10.1287/mnsc.12.7.619

Scherer, B. (2013). Frictional Costs of Diversification. *The Journal of Portfolio Management, 39*(3), 7-9. https://doi.org/10.3905/jpm.2013.39.3.007

Shane, S. (2009). *Fool's Gold? The Truth Behind Angel Investing in America*. Oxford University Press, Oxford, England.

Smidt, S. (1978). Investment Horizons and Performance Measurement. *The Journal of Portfolio Management, 4*(2), Winter, 18-22. https://doi.org/10.3905/jpm.1978.408627

Steinbach, M. (2001). Markowitz Revisited: Mean-Variance Models in Financial Portfolio Analysis. *SIAM Review, 43*(1), (Mar., 2001), 31-85. https://doi.org/10.1137/s0036144500376650

Tarrazo, M. (2018). Mean-Variance under Stress -Ten years of Portfolio Optimizations: 2001-2011. *Quarterly Journal of Finance and Accounting, 50*(1 & 2), 91-148.

Tarrazo, M. (2014). Portfolio Optimization without Programming –Pedagogical and Practical Implications. *Advances in Financial Education, 12*, 90-122.

Tarrazo, M. (2009). Identifying Securities to Buy: The Heuristic ri/stdi. *Research in Finance, 25*, 229-268. https://doi.org/10.1108/s0196-3821(2009)0000025011

Tobin, J. (1958). Liquidity Preference as Behavior Towards Risk. *The Review of Economic Studies, 25*(2), 65-86. https://doi.org/10.2307/2296205

Warschauer, T., & Guerin, A. (1987). Optimal Liquidity in Personal Financial Planning. *The Financial Review, 22*(4), 355-367. https://doi.org/10.1111/j.1540-6288.1987.tb01259.x

Zweig, J. (2022). The Problem with ETSs: Bigger Is Not Better. *The Wall Street Journal*, Saturday-Sunday, January 15-16, B2.